

CLOSURES OF CONJUGACY CLASSES IN CLASSICAL REAL LINEAR LIE GROUPS. II¹

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ABSTRACT. By a classical group we mean one of the groups $GL_n(R)$, $GL_n(C)$, $GL_n(H)$, $U(p, q)$, $O_n(C)$, $O(p, q)$, $SO^*(2n)$, $Sp_{2n}(C)$, $Sp_{2n}(R)$, or $Sp(p, q)$. Let G be a classical group and L its Lie algebra. For each $x \in L$ we determine the closure of the orbit $G \cdot x$ (for the adjoint action of G on L). The problem is first reduced to the case when x is nilpotent. By using the exponential map we also determine the closures of conjugacy classes of G .

0. Introduction. Let G be a classical real linear Lie group (see §1 for precise definition) and L its Lie algebra. There are ten infinite series of such groups and we use index j ($1 \leq j \leq 10$) to label these series. G acts on L via the adjoint representation and on itself by conjugation. The classification of orbits of G in L and the classification of conjugacy classes of G are now well known; see for instance [1] and the references mentioned there. In this paper we determine the closure of an arbitrary orbit \mathcal{O} and the closure of an arbitrary conjugacy class \mathcal{C} . (The topological terms refer to the ordinary topology of G and L .)

For complex groups this problem was solved by M. Gerstenhaber [5, 6]. An independent proof for complex special linear groups was given by J. Dixmier [2], and for complex orthogonal and symplectic groups by W. Hesselink [7] and the author [3]. In fact the results of Gerstenhaber and Hesselink are more general since they consider classical groups over any algebraically closed field.

In our previous paper [4] on the same subject we have announced the main result of the present paper. For the sake of convenience we repeat all the necessary definitions so that the reader does not need to consult [4] except for the proofs of Theorem 1 (§2), Theorem 2 (§3), and the necessity part of Theorem 5 (§7).

The two problems, closures of orbits and closures of conjugacy classes, are closely related. We give a full treatment to the first problem while the second is treated in a cavalier fashion in §13. In this introduction we shall comment only on the first problem.

In §2 we state the Centralizer Theorem (Theorem 1) which asserts that the centralizer $C_G(x)$ in G of a semisimple element $x \in L$ is a direct product of classical groups. The direct factors of $C_G(x)$ may belong to different series of classical groups.

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Each factor is associated with an eigenvalue of x and the type of a factor is uniquely determined by the type of G and the corresponding eigenvalue of x (see Table III). This theorem provides justification and motivation for our definition of classical groups given in §1.

In §3 we show that our problem can be reduced to the case of nilpotent orbits. (An orbit of G in L is nilpotent if it consists of nilpotent elements.) The reduction is based on Theorem 2 for whose proof we refer to our paper [4]. Let $x \in L$ and let $x = x_s + x_n$ be its Jordan decomposition. (This means that $x_s, x_n \in L$, x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$.) Theorem 2 asserts that if y belongs to the closure of the orbit $G \cdot x$ then there exists an element z in the closure of the orbit $C_G(x_s) \cdot x_n$ such that $x_s + z \in G \cdot y$. Hence we can determine the closure of $G \cdot x$ provided we know how to find the closure of the orbit of x_n under the group $C_G(x_s)$. Since $[x_s, x_n] = 0$, the element x_n belongs to the Lie algebra of $C_G(x_s)$. The reduction is completed now by invoking the Centralizer Theorem. Note that if we start with the classical group G of type j (i.e. belonging to the j th series) our reduction requires us to consider the direct factors of $C_G(x_s)$ which may be of different type. For this reason all the classical groups should be treated simultaneously. We have not done so only in order to save space and to avoid duplication of the cases already treated in the literature. Hence in the very short §4 we dismiss the cases $j = 1, 2, 3, 5$ and 8 .

In §5 we recall the definition of types for classical Lie algebras introduced by N. Bourgoynne and R. Cushman [1]. We restrict ourselves to the cases $j = 4, 6, 7, 9, 10$ and the nilpotent types. Let V be a vector space and f a form which defines the classical group G of the j th series (see §1). A type Δ is an equivalence class of triples (V, f, x) where $x \in L$. Each type Δ can be written as a sum of indecomposable types and such a decomposition is unique. For details about types Δ and their properties we refer the reader to [1]. For each indecomposable nilpotent type we give a representative (V, f, x) . This is done mainly because our notation for polarized types Δ^+ differs from the notation in [1] and we wanted to avoid possible confusion. The types Δ_m^+ and Δ_m^- which differ only in the superscript \pm can be distinguished by inequalities (5.3)–(5.5). Their importance in the rest of the paper cannot be overestimated.

In §6 we introduce combinatorial gadgets called chromosomes. They consist of genes which may be polarized or nonpolarized. In fact the set of chromosomes is a free commutative monoid on the set of genes. To each chromosome X we assign its rank $r(X) = n$ and its signature $\text{sig } X = (p, q)$ where p and q are either integers or half-integers such that $p \geq 0$, $q \geq 0$ and $p + q = n$. A submonoid of the set of chromosomes is called a variety. To each series of classical groups we assign a specific variety, say Φ_j to the j th series (see Table IV in §5). We also define a partial order \leq in each of the varieties Φ_j .

In §7 we establish (Lemma 4) a bijection between the nilpotent orbits of the classical group G of the j th series defined by (V, f) and the set of chromosomes X in Φ_j such that $\text{sig } X = \text{sig}(f)$. (See §1 for the definition of $\text{sig}(f)$.) In the sequel we use the chromosomes as convenient labels which parametrize the nilpotent orbits.

We also state here our main result (Theorem 5). It says that a nilpotent orbit Θ_1 is contained in the closure of the nilpotent orbit Θ_2 iff their corresponding chromosomes $X(\Theta_1)$ and $X(\Theta_2)$ satisfy the inequality $X(\Theta_1) \leq X(\Theta_2)$. This latter inequality refers to the partial order in Φ_j defined in §6. In Example 2 we illustrate the main theorem when $G = Sp(2, 2)$ ($j = 10$, $\text{sig}(f) = (2, 2)$). The necessity of the condition $X(\Theta_1) \leq X(\Theta_2)$ was proved in [7] and we do not repeat the proof. The remainder of the paper is in essence the proof of the sufficiency of the above condition. It should be clear that this proof has to be based on the knowledge of the partial order \leq in Φ_j . (This is true in all the cases considered so far in the literature.)

In §8 we describe simple transformations of chromosomes of Φ_j , say $X \rightarrow Y$, such that $X < Y$ and $\text{sig } X = \text{sig } Y$. We call these simple transformations mutations of chromosomes. When $j = 4$ they are given by formulas (8.1)–(8.3), for $j = 7, 10$ by (8.4)–(8.8), and for $j = 6, 9$ by (8.9)–(8.17). Then we state our main theorem about partial orders \leq in Φ_j (Theorem 6). It says that if $X \leq Y$ in Φ_j then there exists a chain of Φ_j -mutations $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m$ ($m \geq 0$) such that $X = X_0$ and $Y = X_m$. This theorem enables us to reduce the proof of Theorem 5 to a finite number of cases.

The proof of (the sufficiency part of) Theorem 5 occupies §§9–12. §9 is of a preliminary nature. §10 treats the case $j = 4$ in detail. In §11 we treat the cases $j = 7, 10$ by reducing them to the case $j = 4$. In §12 we treat the remaining two cases $j = 6, 9$. While in the case $j = 4$ we have included all necessary computations we have not done this in the cases $j = 6, 9$ in order to keep §12 of reasonable size.

As mentioned earlier §13 deals with the problem of closures of conjugacy classes of G . We show that the problem can be reduced to the case of unipotent conjugacy classes and then the exponential map does the job.

The remaining §§14–17 are devoted to the proof of Theorem 6 (see §8). §14 is of a preliminary nature. §15 deals with the case $j = 4$, §16 with the cases $j = 7, 10$, and §17 with the cases $j = 6, 9$.

We use this occasion to correct several misprints in our previous paper [4]. The formula (12.3) should read:

$$g_\epsilon(m) + g_{-\epsilon}(n) \rightarrow g_{-\epsilon}(m-1) + g_\epsilon(n+1), \quad m \leq n.$$

In §9 in the description of the representative (V, f, x) of $\Delta_m^\epsilon(0, 0)$, m odd, $j = 7$ one should have $f(e_r, e_s) = -i\epsilon\delta_{r+s, m}$ where $i = \sqrt{-1}$. Finally on p. 8 line 8 one should replace $x \in L$ by $x \in G$.

1. Classical groups. Let V be a (right) \mathbf{F} -vector space where $\mathbf{F} = \mathbf{R}, \mathbf{C}$, or \mathbf{H} and let $\dim_{\mathbf{F}} V = n < \infty$. (By \mathbf{H} we denote the division algebra of real quaternions.) If $f: V \times V \rightarrow \mathbf{F}$ is a nondegenerate hermitian form (including the case of symmetric forms if $\mathbf{F} = \mathbf{R}$) then we define the signature of f to be the ordered pair $\text{sig}(f) = (n_+, n_-)$ where n_+ (resp. n_-) is the maximum dimension (over \mathbf{F}) of a positive (resp. negative) definite subspace of V . Thus $n_+ + n_- = n$.

When f is a nondegenerate skew-hermitian form (including the case of skew-symmetric forms if $\mathbf{F} = \mathbf{R}$) then we set $\text{sig}(f) = (n/2, n/2)$.

The words *classical group* will have a technical meaning which we now define. There are ten infinite series of classical groups and we use index j ($1 \leq j \leq 10$) to refer to them. The first three series are the general linear groups

$$GL(V) \cong GL_n(\mathbf{F})$$

where $\mathbf{F} = \mathbf{C}$ for $j = 1$, $\mathbf{F} = \mathbf{R}$ for $j = 2$, and $\mathbf{F} = \mathbf{H}$ for $j = 3$.

For the remaining seven series V is equipped with a nondegenerate form f : $V \times V \rightarrow \mathbf{F}$ and the corresponding classical group G is the subgroup of $GL(V)$ which preserves f . The type of the form f is specified in Table I for each of these series.

j	\mathbf{F}	G	f	$\text{sig}(f)$
4	\mathbf{C}	$U(k, n - k)$	hermitian	$(k, n - k)$
5	\mathbf{C}	$O_n(\mathbf{C})$	symmetric	
6	\mathbf{R}	$O(k, n - k)$	symmetric	$(k, n - k)$
7	\mathbf{H}	$O^*(2n)$	skew-hermitian	$(n/2, n/2)$
8	\mathbf{C}	$Sp_{2n}(\mathbf{C})$	skew-symmetric	
9	\mathbf{R}	$Sp_{2n}(\mathbf{R})$	skew-symmetric	(n, n)
10	\mathbf{H}	$Sp(k, n - k)$	hermitian	$(k, n - k)$

TABLE I

Note that, according to our definition, the groups $SL_n(\mathbf{C})$, $SL_n(\mathbf{R})$, $SO_n(\mathbf{C})$, $SO(k, n - k)$, $SU(k, n - k)$, $SL_n(\mathbf{H})$ are not classical.

2. The centralizer theorem. From now on G will denote a classical group belonging to the j th series, L its Lie algebra, and $\text{Ad}: G \rightarrow \text{Aut}(L)$ the adjoint representation of G . For $a \in G$ and $x \in L$ we shall write $a \cdot x$ instead of $\text{Ad}(a)(x)$. Thus $G \cdot x = \{a \cdot x: a \in G\}$ is the G -orbit in L containing x . The *centralizer* $C_G(x)$ of $x \in L$ in G is the subgroup of G consisting of all elements a such that $a \cdot x = x$. In this section we describe the structure of $C_G(x)$ for semisimple x . It turns out that this centralizer is a (finite) direct product of classical groups.

If $1 \leq j \leq 3$ we identify L with the Lie algebra of all \mathbf{F} -linear transformations of V . If $4 \leq j \leq 10$ then we identify L with the Lie algebra of all \mathbf{F} -linear transformations x of V which satisfy $f(x(v_1), v_2) + f(v_1, x(v_2)) = 0$ for all $v_1, v_2 \in V$.

Let $x \in L$ be arbitrary. If $\mathbf{F} = \mathbf{C}$ we define for $\zeta \in \mathbf{C}$ the subspace $V(\zeta, x)$ of V to be

$$V(\zeta, x) = \text{Ker}(x - \zeta)^\infty,$$

i.e., it is the kernel of $(x - \zeta)^k$ for large k , say $k = n$. If $\mathbf{F} = \mathbf{R}$ or \mathbf{H} we define for $\zeta \in \mathbf{C}$ the subspace $V(\zeta, x)$ of V by

$$V(\zeta, x) = \text{Ker}(x^2 - 2 \text{Re}(\zeta)x + |\zeta|^2)^\infty.$$

We say that $\zeta \in \mathbf{C}$ is an *eigenvalue* of $x \in L$ if $V(\zeta, x) \neq 0$.

It is well known that for $j \in \{5, 6\}$ we have $V(\zeta, x) \perp V(\zeta', x)$ if $\zeta + \zeta' \neq 0$, and for $j \in \{4, 6, 7, 9, 10\}$ we have $V(\zeta, x) \perp V(\zeta', x)$ if $\bar{\zeta} + \zeta' \neq 0$. (The orthogonality is with respect to the form f .)

This suggests that we introduce the following definition:

$$V'(\zeta, x) = \begin{cases} V(\zeta, x) & \text{for } j \in \{1, 2, 3\}, \\ V(\zeta, x) + V(-\zeta, x) & \text{for } j \in \{5, 8\}, \\ V(\zeta, x) + v(-\bar{\zeta}, x) & \text{for } j \in \{4, 6, 7, 9, 10\}. \end{cases}$$

Then we obtain a direct (and orthogonal if $j \geq 4$) decomposition

$$(2.1) \quad V = \sum_{\zeta \in \Omega_j} V'(\zeta, x).$$

The set Ω_j is the subset of \mathbf{C} specified by the conditions stated in Table II.

j	Ω_j		
1	\mathbf{C}		
2, 3	$\text{Im } \zeta \geq 0$		
4	$\text{Re } \zeta \geq 0$		
5, 8	$\text{Im } \zeta > 0$	or	$\zeta \geq 0$
6, 7, 9, 10	$\text{Re } \zeta \geq 0$	and	$\text{Im } \zeta \geq 0$

TABLE II

For $\zeta \in \mathbf{C}$ let x_ζ (resp. x'_ζ) be the restriction of x to $V(\zeta, x)$ (resp. $V'(\zeta, x)$). If $1 \leq j \leq 3$ let $G(\zeta, x)$ be the general linear group of the space $V(\zeta, x)$. If $4 \leq j \leq 10$ let $G(\zeta, x)$ be the group of all \mathbf{F} -linear automorphisms of $V'(\zeta, x)$ which preserve the restriction of the form f to $V'(\zeta, x)$. (This restriction is clearly nondegenerate.) Thus in all cases $G(\zeta, x)$ is a classical group of the j th series. If $L(\zeta, x)$ is the Lie algebra of $G(\zeta, x)$ then $x'_\zeta \in L(\zeta, x)$. Finally let $C(\zeta, x)$ be the centralizer of x'_ζ in $G(\zeta, x)$.

Since each of the subspaces $V'(\zeta, x)$ is invariant under $C_G(x)$, we have

$$(2.2) \quad C_G(x) = \prod_{\zeta \in \Omega_j} C(\zeta, x).$$

THEOREM 1. *Let G be a classical group of the j th series, L its Lie algebra, and $x \in L$ a semisimple element. Then each $C(\zeta, x)$ in (2.2) is a classical group of k th series where k depends only on j and ζ .*

For the proof of this theorem we refer to [4].

The values of k are given by Table III (at present, ignore the third column).

j	ξ	λ	k
1	all ξ	all λ	1
2, 3	$\xi = \bar{\xi}$	$\lambda = \bar{\lambda}$	$k = j$
	$\xi \neq \bar{\xi}$	$\lambda \neq \bar{\lambda}$	1
4	$\xi = -\bar{\xi}$	$ \lambda = 1$	4
	$\xi \neq -\bar{\xi}$	$ \lambda \neq 1$	1
$5 \leq j \leq 10$	$\xi = 0$	$\lambda = \pm 1$	$k = j$
5, 8	$\xi \neq 0$	$\lambda \neq \pm 1$	1
6, 7, 9, 10	$\xi \neq \pm \bar{\xi}$	$\lambda \neq \bar{\lambda} \neq \lambda^{-1}$	1
	$\xi = -\bar{\xi} \neq 0$	$\lambda^{-1} = \bar{\lambda} \neq \pm 1$	4
6, 9	$\xi = \bar{\xi} \neq 0$	$\lambda = \bar{\lambda} \neq \pm 1$	2
7, 10	$\xi = \bar{\xi} \neq 0$	$\lambda = \bar{\lambda} \neq \pm 1$	3

TABLE III

3. Reduction to the nilpotent case. In this section we reduce the problem of finding the closure of a G -orbit in L to the case of nilpotent orbits, i.e., orbits consisting of nilpotent elements.

Every $x \in L$ has a unique Jordan decomposition $x = x_s + x_n$, where $x_s \in L$ is semisimple, $x_n \in L$ is nilpotent, and $[x_s, x_n] = 0$. We refer to x_s (resp. x_n) as the *semisimple* (resp. *nilpotent*) *component* of x .

THEOREM 2. *Let G be a classical group, L its Lie algebra, x and y elements of L , and $x = x_s + x_n$ the Jordan decomposition of x . Then $y \in G \cdot x$ iff $a \cdot y = x_s + z$ for some $a \in G$ and some*

$$(3.1) \quad z \in \overline{C_G(x_s) \cdot x_n}.$$

For the proof we refer to [4].

Let \mathcal{O}_1 and \mathcal{O}_2 be G -orbits in L . Assume that $\mathcal{O}_1 \subset \bar{\mathcal{O}}_2$, which we shall also write as $\mathcal{O}_1 \leq \mathcal{O}_2$. By Theorem 2 we can choose $x \in \mathcal{O}_2$ and $y \in \mathcal{O}_1$ such that $y = x_s + z$ and z satisfies (3.1). Clearly, z is the nilpotent part of y , $z = y_n$. By Theorem 1 we have

$$(3.2) \quad C_G(x_s) = C(\xi_1, x_s) \times \cdots \times C(\xi_m, x_s)$$

where ξ_1, \dots, ξ_m are the distinct eigenvalues of x_s belonging to Ω_j .

Let $Z_L(x_s)$ be the centralizer of x_s in L , i.e., $Z_L(x_s) = \{t \in L: [t, x_s] = 0\}$.

We denote by $L(\xi_k, x_s)$ the Lie algebra of $G(\xi_k, x_s)$, and let $Z(\xi_k, x_s)$ be the centralizer in $L(\xi_k, x_s)$ of the restriction of x_s to $V'(\xi_k, x_s)$. It is clear that $Z_L(x_s)$ is the Lie algebra of $C_G(x_s)$, and $Z(\xi_k, x_s)$ is the Lie algebra of $C(\xi_k, x_s)$.

It follows from (3.2) that

$$(3.3) \quad Z_L(x_s) = Z(\zeta_1, x_s) \oplus \cdots \oplus Z(\zeta_m, x_s).$$

Since $x_n, z \in Z_L(x_s)$, we have

$$x_n = x_n^{(1)} + \cdots + x_n^{(m)}, \quad z = z^{(1)} + \cdots + z^{(m)},$$

where $x_n^{(k)}, z^{(k)} \in Z(\zeta_k, x_s)$ for $1 \leq k \leq m$.

It follows from (3.1) to (3.3) that $z^{(k)} \in C(\zeta_k, x_s) \cdot x_n^{(k)}$, $1 \leq k \leq m$.

Thus the G -orbits in L which are contained in the closure of the orbit $G \cdot x$ can be easily found provided we know which orbits of $C(\zeta_k, x_s)$ in $Z(\zeta_k, x_s)$ are contained in the closure of the orbit $C(\zeta_k, x_s) \cdot x_n^{(k)}$, $1 \leq k \leq m$.

Since $C(\zeta_k, x_s)$ is a classical group and $x_n^{(k)} \in Z(\zeta_k, x_s)$ is nilpotent, we have indeed reduced the general problem to the nilpotent case.

Note that (by assumption) G belongs to the j th series but the groups $C(\zeta_k, x_s)$ may belong to some other series of classical groups. This is the reason why we have to consider simultaneously all classical groups.

4. The cases $j = 1, 2, 3, 5, 8$. Recall that for two G -orbits Θ_1 and Θ_2 in L we write $\Theta_1 \leq \Theta_2$ if $\Theta_1 \subset \bar{\Theta}_2$.

The cases $j \in \{1, 5, 8\}$ were treated by M. Gerstenhaber [5, 6]; his results are more general since he works over an arbitrary algebraically closed field (if $j = 5, 8$ the characteristic is $\neq 2$). The case $j = 1$ was also dealt with by J. Dixmier [2]. The cases $j = 2, 3$ can be treated in the similar way as $j = 1$. The cases $j = 5, 8$ were studied in a more general setting by W. Hesselink [7]. All these authors consider only nilpotent G -orbits in L , but their results can be easily extended to arbitrary orbits. Finally the cases $j = 5, 8$ were also considered by the author [3] for arbitrary G -orbits in L as well as for conjugacy classes of G .

The result for nilpotent G -orbits in L can be stated as follows.

THEOREM 3. *Let G be a classical group of j th series with $j \in \{1, 2, 3, 5, 8\}$, and L its Lie algebra. If $x, y \in L$ are nilpotent then $G \cdot x \leq G \cdot y$ holds iff $\text{rank}(x^k) \leq \text{rank}(y^k)$ for all $k (\geq 0)$.*

5. Description of nilpotent orbits. From now on we shall assume that $j \in \{4, 6, 7, 9, 10\}$. For a fixed j we consider the triples (V, f, x) where V and f are specified by Table I and $x \in L$, where L is the Lie algebra of the corresponding classical group G . An equivalence class of such triples is called a *type*; see [1] for precise definition. Let Δ be the type containing the triple (V, f, x) . There exists a direct decomposition $V = V_1 \oplus \cdots \oplus V_r$ into x -invariant nonzero nondegenerate subspaces which are orthogonal to each other and moreover the summands V_1, \dots, V_r cannot be further decomposed in the same sense. Let x_k and f_k be the restrictions of x and f , respectively, to V_k . If Δ_k is the type containing (V_k, f_k, x_k) then we shall write $\Delta = \Delta_1 + \cdots + \Delta_r$. We say that Δ is *indecomposable* if $\Delta \neq 0$ (i.e., $\dim V \neq 0$) and Δ cannot be written as a sum of two nonzero types. In general, Δ can be

expressed as a sum of indecomposable types and as shown in [1] such a decomposition is unique. If x is nilpotent we say that the type Δ is also nilpotent.

The nilpotent indecomposable types are listed in the second column of Table IV. (At present ignore the last two columns of this table.)

j	Δ	X	Φ_j
4	$\Delta_m^\epsilon(0)$	$g^\epsilon(m+1)$	Π
6	$\Delta_m^\epsilon(0), m \text{ even}$	$g^\epsilon(m+1)$	$(2\Lambda, \Pi)$
	$\Delta_m(0, 0), m \text{ odd}$	$2g(m+1)$	
7	$\Delta_m(0, 0), m \text{ even}$	$g(m+1)$	(Π, Λ)
	$\Delta_m^\epsilon(0, 0), m \text{ odd}$	$g^\epsilon(m+1)$	
9	$\Delta_m(0, 0), m \text{ even}$	$2g(m+1)$	$(\Pi, 2\Lambda)$
	$\Delta_m^\epsilon(0), m \text{ odd}$	$g^\epsilon(m+1)$	
10	$\Delta_m^\epsilon(0, 0), m \text{ even}$	$g^\epsilon(m+1)$	(Λ, Π)
	$\Delta_m(0, 0), m \text{ odd}$	$g(m+1)$	

TABLE IV

In this table $\epsilon = \pm$ and $m (\geq 0)$ is an integer.

We shall now describe representative triples (V, f, x) for each of the types in this table. We warn the reader that our designation of types Δ_m^+ and Δ_m^- differs from that adopted in [1].

If $j = 4$ a representative of $\Delta_m^\epsilon(0)$ is given by:

V a complex vector space,

e_0, e_1, \dots, e_m a basis of V ,

$$(5.1) \quad x(e_k) = e_{k+1}i \quad (i = \sqrt{-1}, 0 \leq k < m), \quad x(e_m) = 0,$$

f a hermitian form on V ,

$f(e_r, e_s) = \epsilon \delta_{r+s, m}$ (δ_{ij} Kronecker symbol).

If $j = 6$, a representative of $\Delta_m^\epsilon(0)$, m even, say $m = 2m'$, is given by:

V a real vector space,

e_0, e_1, \dots, e_m a basis of V ,

$$(5.2) \quad \begin{aligned} x(e_k) &= e_{k+1} & (0 \leq k < m'), \\ x(e_k) &= -e_{k+1} & (m' \leq k < m), \end{aligned} \quad x(e_m) = 0,$$

f a symmetric bilinear form on V ,

$f(e_r, e_s) = \epsilon \delta_{r+s, m}$.

If $j = 7$ a representative of $\Delta_m^\epsilon(0, 0)$, m odd, is given by:

V a (right) quaternionic vector space,

e_0, e_1, \dots, e_m a basis of V ,

x given by (5.1),

f a skew-hermitian form on V ,

$f(e_r, e_s) = -i\epsilon \delta_{r+s, m}$.

If $j = 9$ a representative of $\Delta_m^\varepsilon(0)$, m odd, say $m = 2m' - 1$, is given by:

V a real vector space,

e_0, e_1, \dots, e_m a basis of V ,

x given by (5.2),

f a skew-symmetric form on V ,

$f(e_r, e_s) = \varepsilon \delta_{r+s, m}$ for $r < s$.

If $j = 10$ then a representative of $\Delta_m^\varepsilon(0, 0)$, m even, is given by:

V a (right) quaternionic vector space,

e_0, e_1, \dots, e_m a basis of V ,

x given by (5.1),

f a hermitian form on V ,

$f(e_r, e_s) = \varepsilon \delta_{r+s, m}$.

The nilpotent types Δ_m^+ and Δ_m^- can be distinguished by a simple inequality. Indeed, if (V, f, x) is one of the representatives listed above then for all $v \in V$ we have

$$(5.3) \quad (-i)^m \varepsilon f(v, x^m(v)) \geq 0 \quad \text{if } j = 4,$$

$$(5.4) \quad i^m \varepsilon f(v, x^m(v)) \geq 0 \quad \text{if } j = 6, 10,$$

$$(5.5) \quad i^{m-1} \varepsilon f(v, x^m(v)) \geq 0 \quad \text{if } j = 7, 9.$$

If $j = 6$ (resp. 9) a representative of $\Delta_m(0, 0)$, m odd (resp. even) is given by:

V a real vector space,

$e_0, \dots, e_m, e'_0, \dots, e'_m$ a basis of V ,

$$x(e_k) = e_{k+1} \quad (0 \leq k < m), \quad x(e_m) = 0,$$

$$x(e'_k) = -e'_{k+1} \quad (0 \leq k < m), \quad x(e'_m) = 0,$$

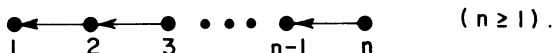
f is a symmetric (resp. skew-symmetric) form on V ,

$$f(e_r, e_s) = f(e'_r, e'_s) = 0, f(e_r, e'_s) = \delta_{r+s, m}.$$

If $j = 7$ then a representative of $\Delta_m(0, 0)$ with m even has the same description as the representative of $\Delta_m^\varepsilon(0, 0)$ with m odd. (The choice of $\varepsilon = \pm$ is not important since both represent the same type.) Similarly, if $j = 10$ then a representative of $\Delta_m(0, 0)$ with m odd has the same description as the representative of $\Delta_m^\varepsilon(0, 0)$ with m even.

6. Chromosomes. In order to state our results about closures of nilpotent orbits we need certain combinatorial gadgets which we call chromosomes. Roughly speaking they are Young diagrams some of whose rows are filled with alternating $+$ and $-$ signs. A precise definition is given below.

A *string* (nonpolarized) S is a directed graph isomorphic to



The vertex 1 is the *head* and the vertex n the *tail* of this string; n is its *rank*. If we assign alternating $+$ and $-$ signs to the vertices of this string we obtain a *polarized string*. A polarized string is *positive* or *negative* depending on whether its tail carries a $+$ or $-$ sign. The *signature* of a polarized string S is the ordered pair

$$(6.1) \quad \text{sig } S = (r^+, r^-)$$

where r^+ and r^- are the numbers of vertices of S with label $+$ and $-$, respectively. If S is nonpolarized we define its signature by (6.1) where now $r^+ = r^- = \frac{1}{2}n$, n being the rank of S .

A *gene* is an isomorphism class of strings; it may be polarized or nonpolarized. There are precisely three genes of rank n for each positive integer n : the nonpolarized gene of rank n , the positive gene of rank n , and the negative gene of rank n . We shall denote these three genes in that order by $g(n)$, $g^+(n)$, and $g^-(n)$.

Let Σ be the free abelian group having the set of genes as its basis; the elements of Σ will be written as formal finite integral linear combinations of genes. A *chromosome* is an element of Σ which is a nonnegative integral linear combination of genes. The set of all chromosomes will be denoted by Σ^+ ; it is clearly a free commutative monoid on the set of genes. If $X = Y + Z$ where Y and Z are chromosomes then we say that X *contains* Y , and we write $X \supset Y$ or $Y \subset X$. If g is a gene and m (≥ 0) integer such that $X \supset mg$ and $X \not\supset (m+1)g$ then we say that m is the *multiplicity* of g in the chromosome X . We say that g *occurs* in X if its multiplicity m in X is not zero. The *signature* of a chromosome X is by definition the sum of the signatures of its genes (taking into account the multiplicities). Similarly, one defines the *rank* of X .

A chromosome is called *even* (resp. *odd*, *polarized*, *nonpolarized*) if all its constituent genes have even rank (resp. have odd rank, are polarized, are nonpolarized). Every chromosome X admits a unique decomposition $X = e(X) + o(X)$ where $e(X)$ is an even and $o(X)$ an odd chromosome.

A *variety* is a submonoid of Σ^+ . By Π (resp. Λ) we denote the variety of polarized (resp. nonpolarized) chromosomes. If Φ is a variety, so is $k\Phi = \{kX: X \in \Phi\}$, for every nonnegative integer k . If Φ is a variety and n (≥ 0) an integer then $\Phi(n)$ denotes the set of all $X \in \Phi$ having rank n . If Φ and Ψ are varieties then (Φ, Ψ) denotes the variety consisting of all chromosomes X such that $e(X) \in \Phi$ and $o(X) \in \Psi$.

We shall need in the sequel the following five varieties:

$$(6.2) \quad \Pi, (\Lambda, \Pi), (\Pi, \Lambda), (2\Lambda, \Pi), (\Pi, 2\Lambda).$$

If g is a gene of rank n (> 1) we define g' as follows

$$\begin{aligned} g' &= g(n-1) && \text{if } g = g(n), \\ &= g^+(n-1) && \text{if } g = g^+(n), \\ &= g^-(n-1) && \text{if } g = g^-(n). \end{aligned}$$

If g is a gene of rank 1 we set $g' = 0$. If X is a chromosome then we define X' by applying this "prime" operation to each of its constituent genes. Now we define inductively the chromosome $X^{(k)}$, $k \geq 0$, as follows:

$$X^{(0)} = X, \quad X^{(k+1)} = (X^{(k)})' \quad \text{for } k \geq 0.$$

For instance, if $X = 2g(3) + g^-(2) + g(2) + g^+(4)$ then

$$X' = X^{(1)} = 2g(2) + g^-(1) + g(1) + g^+(3),$$

$$X^{(2)} = 2g(1) + g^+(2), \quad X^{(3)} = g^+(1), \quad X^{(k)} = 0 \quad \text{for } k \geq 4.$$

For ordered pairs of real numbers, the inequality $(p, q) \leq (r, s)$ will stand for $p \leq r$ and $q \leq s$. We say that a chromosome X *dominates* a chromosome Y , and we write $X \geq Y$ or $Y \leq X$, if

$$\text{sig } X^{(k)} \geq \text{sig } Y^{(k)}, \quad \forall k.$$

It is easy to check that in each of the varieties (6.2) the dominance relation is a partial order.

7. Statement of the main result. To each indecomposable nilpotent type Δ of the j th series we associate a chromosome $X = X(\Delta)$ belonging to a variety Φ_j . The chromosome X and the variety Φ_j are specified in the last two columns of Table IV.

More generally if Δ is a nilpotent type of the j th series and $\Delta = \Delta_1 + \cdots + \Delta_r$ is its decomposition into indecomposable types then we define $X(\Delta) = X(\Delta_1) + \cdots + X(\Delta_r)$.

Clearly the map $\Delta \mapsto X(\Delta)$ from the set of nilpotent types of the j th series to Φ_j is a bijection.

Now let V, \mathbf{F}, f and G be as in §1, L the Lie algebra of G and $n = \dim_{\mathbf{F}} V$. Since $j \in \{4, 6, 7, 9, 10\}$, the signature $\text{sig}(f)$ was defined in §1. When $j = 7$ or 9 we have $\text{sig}(f) = (n/2, n/2)$.

Let \mathcal{O} be a nilpotent G -orbit in L . If $x, y \in \mathcal{O}$ then the triples (V, f, x) and (V, f, y) are equivalent and hence we can associate a type $\Delta = \Delta(\mathcal{O})$ to the orbit \mathcal{O} . We shall write $X(\mathcal{O})$ instead of $X(\Delta(\mathcal{O}))$.

The following lemma is an immediate consequence of our definitions and the description of nilpotent G -orbits in L given in §5.

LEMMA 4. *The map $\mathcal{O} \mapsto X(\mathcal{O})$ defined above is a bijection from the set of nilpotent G -orbits in L to the set of chromosomes $X \in \Phi_j$ satisfying the condition $\text{sig } X = \text{sig}(f)$.*

EXAMPLE 1. Let $j = 10$ and $\text{sig}(f) = (2, 2)$. Then there are six nilpotent orbits in L ; their types are

$$\begin{aligned} 2\Delta_0^+(0, 0) + 2\Delta_0^-(0, 0), \quad \Delta_0^+(0, 0) + \Delta_0^-(0, 0) + \Delta_1(0, 0), \quad 2\Delta_1(0, 0), \\ \Delta_0^+(0, 0) + \Delta_2^-(0, 0), \quad \Delta_0^-(0, 0) + \Delta_2^+(0, 0), \quad \Delta_3(0, 0). \end{aligned}$$

These types are also listed in [1, p. 351] but the superscripts \pm do not coincide due to the different definition which we have introduced.

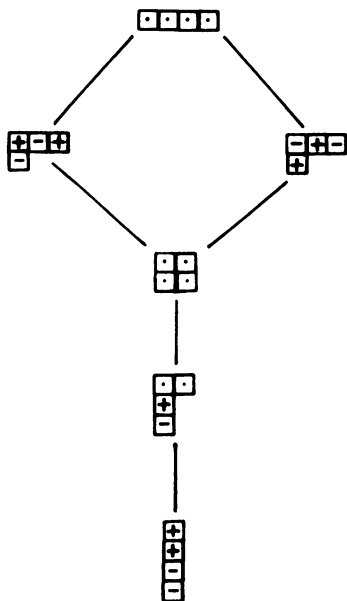
The chromosomes in (Λ, Π) corresponding to these types are $2g^+(1) + 2g^-(1)$, $g^+(1) + g^-(1) + g(2)$, $2g(2)$, $g^+(1) + g^-(3)$, $g^-(1) + g^+(3)$, and $g(4)$, respectively.

We can now state our main result. Recall that we assume that G is a classical group of the j th series with $j \in \{4, 6, 7, 9, 10\}$.

THEOREM 5. *If \mathcal{O}_1 and \mathcal{O}_2 are nilpotent orbits of G in its Lie algebra L then $\mathcal{O}_1 \subset \bar{\mathcal{O}}_2$ holds iff $X(\mathcal{O}_1) \leq X(\mathcal{O}_2)$.*

The necessity of the condition $X(\mathcal{O}_1) \leq X(\mathcal{O}_2)$ was proved in [4]. Our main goal in the remaining part of this paper is to prove the sufficiency of this condition.

EXAMPLE 2. As we saw in Example 1, the group $Sp(2, 2)$ has 6 nilpotent orbits in its Lie algebra. The partially ordered set of these orbits (with respect to the ordering \leq) is represented by the diagram



where the orbits are represented by their chromosomes and the rows of dots represent nonpolarized genes. Of course the bottom chromosome $2g^+(1) + 2g^-(1)$ represents the trivial orbit consisting of the zero element only.

8. Mutations. The proof of the sufficiency part of Theorem 5 is based on some combinatorial results which we state in this section. The proofs of these results are postponed until the end of the paper.

For each variety Φ in (6.2) we shall define some transformations of chromosomes which we call Φ -mutations. A chromosome $X \in \Phi$ is transformed by a Φ -mutation into another chromosome $Y \in \Phi$ such that $X \leq Y$, $X \neq Y$, and $\text{sig } X = \text{sig } Y$. We shall indicate such a mutation symbolically by $X \rightarrow Y$. We warn the reader that we may have two Φ -mutations $X \rightarrow Y$ and $X \rightarrow Z$ with $Y \neq Z$ (see the example below).

If $X \rightarrow Y$ is a Φ -mutation and $Z \in \Phi$ then also $X + Z \rightarrow Y + Z$ is a Φ -mutation. Conversely, if $X, Y, Z \in \Phi$ and $X + Z \rightarrow Y + Z$ is a Φ -mutation then also $X \rightarrow Y$ is a Φ -mutation. Hence it suffices to list only the Φ -mutations $X \rightarrow Y$ such that X and Y are *disjoint*, i.e., X and Y have no genes in common. Such Φ -mutations we call *primitive*.

The primitive Π -mutations are the following:

$$(8.1) \quad g^\epsilon(m) + g^{-\epsilon}(n) \rightarrow g^{-\epsilon}(m-1) + g^\epsilon(n+1), \quad m \leq n;$$

$$(8.2) \quad g^\epsilon(m) + g^\epsilon(n) \rightarrow g^\epsilon(m-2) + g^\epsilon(n+2), \quad 1 < m \leq n;$$

$$(8.3) \quad g_\epsilon(m) + g_{-\epsilon}(n) \rightarrow g_{-\epsilon}(m-1) + g_\epsilon(n+1), \quad m \leq n;$$

where $g_\epsilon(n) = g^{(-1)^{n-1}\epsilon}(n)$, $\epsilon = \pm$.

EXAMPLE 3. If $X = g^+(5) + g^+(4) + g^-(1)$ and $X \rightarrow Y$ is a Π -mutation then Y is one of the chromosomes $Y_1 = g^-(6) + g^+(4)$, $Y_2 = g^+(5) + g^-(5)$, $Y_3 = g^+(6) + g^+(3) + g^-(1)$, or $Y_4 = g^+(6) + g^+(4)$. The mutations $X \rightarrow Y_1$ and $X \rightarrow Y_2$ are of type (8.1) while the mutations $X \rightarrow Y_3$ and $X \rightarrow Y_4$ are of type (8.3). (Since X has three genes, none of these mutations is primitive.)

If $\Phi = (\Lambda, \Pi)$ or (Π, Λ) then the primitive Φ -mutations are the following:

$$(8.4) \quad g(m) + g(n) \rightarrow g^\epsilon(m-1) + g^{-\epsilon}(n+1), \quad m \leq n;$$

$$(8.5) \quad g(m) + g^\epsilon(n) \rightarrow g^\epsilon(m-1) + g(n+1), \quad m < n;$$

$$(8.6) \quad g^\epsilon(m) + g(n) \rightarrow g(m-1) + g^\epsilon(n+1), \quad m < n;$$

$$(8.7) \quad g^\epsilon(m) + g^{-\epsilon}(n) \rightarrow g(m-1) + g(n+1), \quad m \leq n;$$

$$(8.8) \quad g^\epsilon(m) + g^\epsilon(n) \rightarrow g^\epsilon(m-2) + g^\epsilon(n+2), \quad 1 < m \leq n;$$

where $\epsilon = \pm$. It should be understood that if, say, $\Phi = (\Lambda, \Pi)$ then in (8.4) m and n are even positive integers, etc.

Finally if $\Phi = (2\Lambda, \Pi)$ or $(\Pi, 2\Lambda)$ then the primitive Φ -mutations are the following:

$$(8.9) \quad 2g(m) \rightarrow g^\epsilon(m-1) + g^{-\epsilon}(m+1);$$

$$(8.10) \quad g^\epsilon(m) + g^{\epsilon'}(n) \rightarrow g^\epsilon(m-2) + g^{\epsilon'}(n+2), \quad 1 < m \leq n;$$

$$(8.11) \quad g^\epsilon(m) + g^+(n) + g^-(n) \rightarrow g^\epsilon(m-2) + 2g(n+1), \quad 1 < m \leq n;$$

$$(8.12) \quad g^+(m) + g^-(m) + g^\epsilon(n) \rightarrow 2g(m-1) + g^\epsilon(n+2), \quad m \leq n;$$

$$(8.13) \quad g^+(m) + g^-(m) + g^+(n) + g^-(n) \rightarrow 2g(m-1) + 2g(n+1), \quad m \leq n;$$

$$(8.14) \quad 2g^\epsilon(m) + 2g^{-\epsilon}(n) \rightarrow 2g(m-1) + 2g(n+1), \quad m \leq n;$$

$$(8.15) \quad g^\epsilon(m) + g^{-\epsilon}(n) \rightarrow g^{-\epsilon}(m-2) + g^\epsilon(n+2), \quad 1 < m \leq n;$$

$$(8.16) \quad 2g^\epsilon(m) + g^{-\epsilon}(n) \rightarrow 2g(m-1) + g^\epsilon(n+2), \quad m \leq n;$$

$$(8.17) \quad g^\epsilon(m) + 2g^{-\epsilon}(n) \rightarrow g^{-\epsilon}(m-2) + 2g(n+1), \quad 1 < m \leq n;$$

where $\epsilon, \epsilon' = \pm$. It should be also understood that $g(0)$ and $g^\epsilon(0)$, $\epsilon = \pm$, are zero chromosomes.

This completes the definition of Φ -mutations when Φ is one of the varieties (6.2).

We say that *there are enough Φ -mutations* if given $X, Y \in \Phi$, such that $X \leq Y$ and $\text{sig } X = \text{sig } Y$, there exists a finite chain of Φ -mutations

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_m \quad (m \geq 0)$$

with $X_0 = X$ and $X_m = Y$.

The combinatorial result which we need is the following.

THEOREM 6. *If Φ is one of the varieties (6.2) then there are enough Φ -mutations.*

The proof of this theorem is given in §§14–17. It is much harder than the proof of the analogous results for the varieties Λ , $(2\Lambda, \Lambda)$, and $(\Lambda, 2\Lambda)$ which are given in [3] and [7].

9. Sufficiency part of Theorem 5. Recall that G is a classical group of the j th series defined by a pair (V, f) and $j \in \{4, 6, 7, 9, 10\}$. Let \mathcal{O}_1 and \mathcal{O}_2 be two nilpotent orbits of G in its Lie algebra L . Let $X_1 = X(\mathcal{O}_1)$ and $X_2 = X(\mathcal{O}_2)$ be the corresponding chromosomes in Φ_j . By hypothesis we have $X_1 \leq X_2$. We also know that $\text{sig } X_1 = \text{sig } X_2 = \text{sig}(f)$. We have to show that $\mathcal{O}_1 \leq \mathcal{O}_2$, i.e., $\mathcal{O}_1 \subset \bar{\mathcal{O}}_2$. By Theorem 6 it suffices to consider the case when

$$(9.1) \quad X_1 \rightarrow X_2$$

is a Φ_j -mutation. Without any loss of generality we may (and we will) assume that this mutation is primitive.

In order to prove that $\mathcal{O}_1 \leq \mathcal{O}_2$ we shall construct a linear one-parameter family

$$(9.2) \quad x(t) = x_0 + tx_1$$

in L such that $x_0 \in \mathcal{O}_1$ and $x(t) \in \mathcal{O}_2$ for $t > 0$. (Only in one case we shall use a quadratic family.) Indeed the existence of such a family implies that $x_0 \in \bar{\mathcal{O}}_2$, and so $\mathcal{O}_1 \subset \bar{\mathcal{O}}_2$.

In order to verify that, say, $x_0 \in \mathcal{O}_1$ it suffices to show that $(V, f, x_0) \in \Delta(\mathcal{O}_1)$. For this purpose we decompose V into an orthogonal direct sum of x_0 -invariant subspaces V_1, \dots, V_k so that the corresponding types $\Delta_1, \dots, \Delta_k$ are indecomposable. We identify these indecomposable types by the criteria of §5 and verify that $\Delta(\mathcal{O}_1) = \Delta_1 + \dots + \Delta_k$. The same method is used to prove that $x(t) \in \mathcal{O}_2$ for $t > 0$.

We shall distinguish several cases according to the value of j .

10. The case $j = 4$. We have $\Phi_4 = \Pi$ and the primitive Π -mutations are defined by (8.1)–(8.3). These mutations can also be written as

$$(10.1) \quad g^\varepsilon(m+1) + g^{\varepsilon'}(n+1) \rightarrow g^{\varepsilon'}(m) + g^\varepsilon(n+2), \quad n-m = 2k+1;$$

$$(10.2) \quad g^\varepsilon(m+1) + g^\varepsilon(n+1) \rightarrow g^\varepsilon(m-1) + g^\varepsilon(n+3), \quad m \leq n;$$

$$(10.3) \quad g^{-\varepsilon}(m+1) + g^\varepsilon(n+1) \rightarrow g^{-\varepsilon'}(m) + g^{\varepsilon'}(n+2), \quad n-m = 2k;$$

where $k (\geq 0)$ is an integer and $\varepsilon, \varepsilon' = \pm$.

Indeed, (8.2) and (10.2) list the same mutations. Mutations (8.1) for $n-m$ even are the same as (10.3) for $\varepsilon' = \varepsilon$. Mutations (8.1) for $n-m$ odd are the same as (10.1) for $\varepsilon' = -\varepsilon$. Mutations (8.3) for $n-m$ even are the same as (10.3) for $\varepsilon' = \varepsilon$. Finally, mutations (8.3) for $n-m$ odd are the same as (10.1) for $\varepsilon' = \varepsilon$.

Assume first that the mutation (9.1) is of the form (10.1). We can choose a basis $e_0, \dots, e_m, e'_0, \dots, e'_n$ of V such that

$$(10.4) \quad f(e_r, e'_s) = 0, \quad f(e_r, e_s) = \varepsilon \delta_{r+s, m}, \quad f(e'_r, e'_s) = \varepsilon' \delta_{r+s, n}$$

holds for all relevant r, s . We define linear maps x_0 and x_1 on V by specifying their effect on basis elements:

$$\begin{aligned} x_0 e_r &= i e_{r+1} & (0 \leq r < m), & \quad x_0 e_m = 0, \\ x_0 e'_r &= i e'_{r+1} & (0 \leq r < n), & \quad x_0 e'_n = 0, \end{aligned}$$

$$\begin{aligned} x_1 e_r &= 0 \quad (0 \leq r < m), \quad x_1 e_m = i e'_{n-k}, \\ x_1 e'_r &= 0 \quad \text{for } r \neq k, \quad x_1 e'_k = i \varepsilon' e_0. \end{aligned}$$

In view of (5.1) and the definition of x_0 it is clear that $x_0 \in \mathcal{O}_1$. It is easy to check that $x_1 \in L$ and so $x(t) \in L$ for all real t . We have

$$\begin{aligned} x(t)^{k+1} e'_0 &= x(t) i^k e'_k = i^{k+1} (e'_{k+1} + \varepsilon \varepsilon' t e_0), \\ x(t)^{k+m+1} e'_0 &= i^{k+m+1} (e'_{k+m+1} + \varepsilon \varepsilon' t e_m) = i^{n-k} (e'_{n-k} + \varepsilon \varepsilon' t e_m), \\ x(t)^n e'_0 &= i^{n-k} x(t)^k (e'_{n-k} + \varepsilon \varepsilon' t e_m) = i^n (e'_n + \varepsilon \varepsilon' t^2 e'_{n-1}), \\ x(t)^{n+1} e'_0 &= i^{n+1} \varepsilon \varepsilon' t^2 e'_n, \quad x(t)^{n+2} e'_0 = 0, \\ x(t)^{m-1} (e_1 - t e'_{k+1}) &= i^{m-1} (e_m - t e'_{k+m}) = i^{m-1} (e_m - t e'_{n-k-1}), \\ x(t)^m (e_1 - t e'_{k+1}) &= 0. \end{aligned}$$

We now assume that $t \neq 0$. Let V_1 and V_2 be the $x(t)$ -invariant subspaces generated by the vectors e'_0 and $e = e_1 - t e'_{k+1}$, respectively. It is clear that $V = V_1 \oplus V_2$, $\dim V_1 = n + 2$, $\dim V_2 = m$. Note that the vectors $e_r - t e'_{r+k}$ ($1 \leq r \leq m$) form a basis of V_2 and are orthogonal to e'_0 . Hence $V_1 \cap V_2^\perp$ is an $x(t)$ -invariant subspace of V_1 containing e'_0 . It follows from the definition of V_1 that $V_1 \cap V_2^\perp = V_1$, i.e., $V_1 = V_2^\perp$. Let Δ_1 and Δ_2 be the types which correspond to V_1 and V_2 , respectively. Since

$$\varepsilon(-i)^{n+1} f(e'_0, x(t)^{n+1} e'_0) = t^2 = \varepsilon'(-i)^{m-1} f(e, x(t)^{m-1} e),$$

it follows from (5.3) that

$$\Delta_1 = \Delta_{n+1}^\varepsilon(0), \quad \Delta_2 = \Delta_{m-1}^{\varepsilon'}(0).$$

Hence indeed $\Delta_1 + \Delta_2 = \Delta(\mathcal{O}_2)$, and so $x(t) \in \mathcal{O}_2$ for $t \neq 0$.

Next assume that the mutation (9.1) is of the form (10.2). Then we can choose a basis $e_0, \dots, e_m, e'_0, \dots, e'_n$ of V such that (10.4) holds with $\varepsilon' = \varepsilon$. We define $x(t)$ by

$$\begin{aligned} x(t) e_0 &= i(e_1 + t e'_0), \quad x(t) e_r = i e_{r+1} \quad (0 < r < m), \quad x(t) e_m = 0, \\ x(t) e'_r &= i e'_{r+1} \quad (0 \leq r < n), \quad x(t) e'_n = i t e_m. \end{aligned}$$

As in the previous case we have $x_0 \in \mathcal{O}_1$ and $x_1 \in L$. We have

$$\begin{aligned} x(t)^m e_0 &= i x(t)^{m-1} (e_1 + t e'_0) = i^m (e_m + t e'_{m-1}), \\ x(t)^{n+2} e_0 &= i^m x(t)^{n-m+2} t e'_{m-1} = i^{n+1} t x(t) e'_n = i^{n+2} t^2 e_m, \\ x(t)^{n+3} e_0 &= 0. \end{aligned}$$

If $m \neq 1$ we have

$$x(t)^{m-2} (t e_1 - e'_{n-m+2}) = i^{m-2} (t e_{m-1} - e'_n), \quad x(t)^{m-1} (t e_1 - e'_{n-m+2}) = 0.$$

Let $t \neq 0$ and let V_1 be the $x(t)$ -invariant subspace generated by e_0 . Clearly, $\dim V_1 = n + 3$ and so if $m = 1$ we have $V_1 = V$. When $m \neq 1$ let V_2 be the $x(t)$ -invariant subspace generated by the vector $e = t e_1 - e'_{n-m+2}$. Clearly, $\dim V_2 = m - 1$ and $V = V_1 \oplus V_2$. The vectors $t e_r - e'_{n-m+r+1}$ ($1 \leq r \leq m - 1$) form a basis of V_2 and are orthogonal to e_0 . As in the previous case this implies that

$V_1 = V_2^\perp$. Let Δ_1 be the type associated with V_1 and if $m \neq 1$ let Δ_2 be the type associated with V_2 . Since

$$\varepsilon(-i)^{n+2} f(e_0, x(t)^{n+2} e_0) = t^2,$$

and if $m \neq 1$

$$\varepsilon(-i)^{m-2} f(e, x(t)^{m-2} e) = t^2,$$

we have $\Delta_1 = \Delta_{n+2}^\varepsilon(0)$, $\Delta_2 = \Delta_{m-2}^\varepsilon(0)$ ($m \neq 1$). Thus $\Delta_1 = \Delta(\mathcal{O}_2)$ if $m = 1$ and $\Delta_1 + \Delta_2 = \Delta(\mathcal{O}_2)$ if $m \neq 1$. Therefore $x(t) \in \mathcal{O}_2$ for $t \neq 0$.

Finally assume that the mutation (9.1) is of the form (10.3). Then we can choose a basis $e_0, \dots, e_{m+k}, e'_0, \dots, e'_{m+k}$ of V such that

$$(10.5) \quad f(e_r, e_s) = f(e'_r, e'_s) = 0, \quad f(e_r, e'_s) = \delta_{r+s, m+k}$$

holds for all r, s . We define linear operators x_0 and x_1 on V by

$$\begin{aligned} x_0 e_r &= i e_{r+1} \quad \text{for } r \neq k-1, m+k, \\ x_0 e_{k-1} &= i(e_k + \varepsilon e'_0) \quad \text{if } k > 0, \\ x_0 e_{m+k} &= i \varepsilon e'_{m+1} \quad (e'_{m+1} = 0 \text{ if } k = 0), \\ x_0 e'_r &= i e'_{r+1} \quad (0 \leq r < m+k), \quad x_0 e'_{m+k} = 0; \\ x_1 e_r &= 0 \quad \text{for } r \neq k, m+k, \\ x_1 e_k &= i \varepsilon e'_0, \quad x_1 e_{m+k} = i \varepsilon e'_m, \\ x_1 e'_r &= 0 \quad (0 \leq r \leq m+k). \end{aligned}$$

From the definition of x_0 and from (10.5) it follows that $f(x_0 e_r, e_s) = 0 = f(e_r, x_0 e_s)$ for all r, s except (when $k > 0$) that

$$\begin{aligned} f(x_0 e_{k-1}, e_{m+k}) &= -i \varepsilon = f(x_0 e_{m+k}, e_{k-1}), \\ f(e_{k-1}, x_0 e_{m+k}) &= i \varepsilon = f(e_{m+k}, x_0 e_{k-1}). \end{aligned}$$

Hence for all r, s we have $f(x_0 e_r, e_s) + f(e_r, x_0 e_s) = 0$.

Since the subspace spanned by e'_0, \dots, e'_{m+k} is totally isotropic it is clear that $f(x_0 e'_r, e'_s) + f(e'_r, x_0 e'_s) = 0$ holds for all r, s .

We have $f(x_0 e_r, e'_s) = 0 = f(e_r, x_0 e'_s)$ for all r, s except that

$$\begin{aligned} f(x_0 e_r, e'_{m+k-r-1}) &= i \quad (0 \leq r < m+k), \\ f(e_r, x_0 e'_{m+k-r-1}) &= -i \quad (0 \leq r < m+k). \end{aligned}$$

Thus for all r, s we have $f(x_0 e_r, e'_s) + f(e_r, x_0 e'_s) = 0$.

Since f is a hermitian form, it follows that $f(x_0 v_1, v_2) + f(v_1, x_0 v_2) = 0$ for all $v_1, v_2 \in V$, i.e., $x_0 \in L$.

It is obvious that $f(x_1 e_r, e'_s) + f(e_r, x_1 e'_s) = 0$ and $f(x_1 e'_r, e'_s) + f(e'_r, x_1 e'_s) = 0$ for all r, s . We have $f(x_1 e_r, e_s) = 0 = f(e_r, x_1 e_s)$ for all r, s except for

$$\begin{aligned} f(x_1 e_k, e_{m+k}) &= -i \varepsilon' = f(x_1 e_{m+k}, e_k), \\ f(e_k, x_1 e_{m+k}) &= i \varepsilon' = f(e_{m+k}, x_1 e_k). \end{aligned}$$

Hence $f(x_1 e_r, e_s) + f(e_r, x_1 e_s) = 0$ for all r, s . Therefore $x_1 \in L$ and $x(t) = x_0 + t x_1 \in L$ for all real t .

We claim that $x_0 \in \mathcal{O}_1$. Let first $k = 0$, i.e., $m = n$. For real λ let U_λ be the x_0 -invariant subspace generated by the vector $e_0 + \lambda e'_0$. The vectors $e_r + \lambda e'_r$ ($0 \leq r \leq m$) form a basis of U_λ . Since

$$(10.6) \quad \varepsilon(-i)^m f(e_0 + \lambda e'_0, x_0^m(e_0 + \lambda e'_0)) = 2\varepsilon\lambda,$$

U_λ is nondegenerate for $\lambda \neq 0$. (Indeed, if degenerate then $x_0^m(e_0 + \lambda e'_0)$ would be in its radical.) Since $e_0 + e'_0$ is orthogonal to $e_r - e'_r$ ($0 \leq r \leq m$), it follows that $V = U_1 \oplus U_{-1}$ and $U_1 \perp U_{-1}$. From (10.6) and (5.3) it follows now that $(V, f, x_0) \in \Delta_m^+(0) + \Delta_m^-(0)$, and so $x_0 \in \mathcal{O}_1$. Now let $k > 0$. We have then

$$\begin{aligned} x_0^n e_0 &= i^k x_0^{n-k}(e_k + \varepsilon e'_0) = i^{m+k} x_0^k(e_{m+k} + \varepsilon e'_m) = 2\varepsilon i^n e'_{m+k}, \\ x_0^{n+1} e_0 &= 0, \quad x_0^m(e_k - \varepsilon e'_0) = i^m(e_{m+k} - \varepsilon e'_m), \\ x_0^{m+1}(e_k - \varepsilon e'_0) &= 0. \end{aligned}$$

Let V_1 and V_2 be the x_0 -invariant subspaces generated by e_0 and $e = e_k - \varepsilon e'_0$, respectively. Since e'_{m+k} and $e_k - \varepsilon e_0$ are linearly independent vectors in $\text{Ker } x_0$, the above equalities imply that $V = V_1 \oplus V_2$, $\dim V_1 = n + 1$ and $\dim V_2 = m + 1$. Since the vectors $e_{k+r} - \varepsilon e'_r$ ($0 \leq r \leq m$) form a basis of V_2 and are orthogonal to e_0 , we have $V_1 \perp V_2$. Since $\varepsilon(-i)^n f(e_0, x_0^n e_0) = 2 = -\varepsilon(-i)^m f(e, x_0^m e)$, it follows from §5 that $(V, f, x_0) \in \Delta_m^\varepsilon(0) + \Delta_n^\varepsilon(0)$. Thus again $x_0 \in \mathcal{O}_1$.

For $k > 0$ we have

$$\begin{aligned} x(t)^{n+1} e_0 &= i^k x(t)^{n-k+1}(e_k + \varepsilon e'_0) = i^{k+1} x(t)^{n-k}(e_{k+1} + \varepsilon' t e'_0 + \varepsilon e'_1) \\ &= i^{m+k} x(t)^{k+1}(e_{m+k} + \varepsilon' t e'_{m-1} + \varepsilon e'_m) \\ &= 2i^{m+k+1} x(t)^k(\varepsilon' t e'_m + \varepsilon e'_{m+1}) = 2i^{n+1} \varepsilon' t e'_{m+k}, \end{aligned}$$

and for $k = 0$

$$x(t)^{n+1} e_0 = i x(t)^n(e_1 + \varepsilon' t e'_0) = i^n x(t)(e_n + \varepsilon' t e'_{n-1}) = 2i^{n+1} \varepsilon' t e'_n.$$

Thus for all k we have

$$(10.7) \quad x(t)^{n+1} e_0 = 2i^{n+1} \varepsilon' t e'_{m+k}, \quad x(t)^{n+2} e_0 = 0.$$

Writing $a = e_{k+1} - \varepsilon' t e'_0 - \varepsilon e'_1$, we have

$$(10.8) \quad x(t)^{m-1} a = i^{m-1}(e_{m+k} - \varepsilon' t e'_{m-1} - \varepsilon e'_m), \quad x(t)^m a = 0.$$

Fix $t \neq 0$ and let W_1 and W_2 be $x(t)$ -invariant subspaces generated by e_0 and a , respectively. Let first $k > 0$. Then the vectors $e_{r+k} - \varepsilon' t e'_{r-1} - \varepsilon e'_r$ ($1 \leq r \leq m$) form a basis of W_2 and are orthogonal to e_0 . From (10.7) and (10.8) we conclude that $V = W_1 \oplus W_2$, $\dim W_1 = n + 2$, $\dim W_2 = m$, and $W_1 \perp W_2$. Now let $k = 0$ and let W_3 be the $x(t)$ -invariant subspace generated by the vector $b = e_1 - \varepsilon' t e'_0$. Then $x(t)^{m-1} b = i^{m-1}(e_m - \varepsilon' t e'_{m-1})$, $x(t)^m b = 0$. Hence the vectors $e_r - \varepsilon' t e'_{r-1}$ ($1 \leq r \leq m$) form a basis of W_3 and are orthogonal to e_0 . It follows that $V = W_1 \oplus W_3$ and $W_1 \perp W_3$.

Since

$$\begin{aligned}\epsilon'(-i)^{n+1}f(e_0, x(t)^{n+1}e_0) &= 2t, \\ -\epsilon'(-i)^{m-1}f(a, x(t)^{m-1}a) &= 2t, \\ -\epsilon'(-i)^{m-1}f(b, x(t)^{m-1}a) &= 2t \quad (k=0),\end{aligned}$$

it follows that $x(t) \in \mathcal{O}_2$ for $t > 0$.

11. The cases $j = 7, 10$. We shall reduce these cases to the previous case $j = 4$. Recall that $\Phi_7 = (\Pi, \Lambda)$ and $\Phi_{10} = (\Lambda, \Pi)$.

Given $X \in \Pi$ we shall denote by $\sigma(X)$, resp. $\tau(X)$, the chromosome in (Π, Λ) , resp. (Λ, Π) , which is obtained from X by ignoring the polarization of its odd, resp. even, genes. Clearly $\sigma: \Pi \rightarrow (\Pi, \Lambda)$ and $\tau: \Pi \rightarrow (\Lambda, \Pi)$ are morphisms of monoids.

LEMMA 7. *Every primitive (Π, Λ) -mutation, resp. (Λ, Π) -mutation, is of the form $\sigma(X) \rightarrow \sigma(Y)$, resp. $\tau(X) \rightarrow \tau(Y)$, for a suitable primitive Π -mutation $X \rightarrow Y$.*

PROOF. We have to show that each of the mutations (8.4)–(8.8) can be obtained by applying σ or τ to a primitive Π -mutation $X \rightarrow Y$. In the case (8.4) we take $X = g^{-\epsilon}(m) + g^{\epsilon}(n)$, Y is determined by (8.4). In the case (8.5) we take $X = g^{-\epsilon}(m) + g^{\epsilon}(n)$, $Y = g^{\epsilon}(m-1) + g^{-\epsilon}(n+1)$. In the case (8.6) we take $X = g^{\epsilon}(m) + g^{-\epsilon}(n)$, $Y = g^{-\epsilon}(m-1) + g^{\epsilon}(n+1)$. In the case (8.7) we take $Y = g^{-\epsilon}(m-1) + g^{\epsilon}(n+1)$, X being already determined by (8.7). The mutations (8.8) are also primitive Π -mutations.

Let U be a classical group of the fourth series defined by a pair (W, g) . Let $W' = W \otimes_{\mathbb{C}} \mathbf{H}$ and let g_{σ} , resp. g_{τ} , be the skew-hermitian, resp. hermitian, form on W' whose restriction to W is $-ig$, resp. g . Since g is nondegenerate, so are g_{σ} and g_{τ} . Let U_{σ} , resp. U_{τ} , be the classical group of the seventh, resp. tenth, series defined by the pair (W', g_{σ}) , resp. (W', g_{τ}) . Clearly we may view U as a subgroup of U_{σ} and U_{τ} . Let M (resp. M_{σ} , M_{τ}) be the Lie algebra of U (resp. U_{σ} , U_{τ}). We shall view M as a subalgebra of M_{σ} and M_{τ} . Let $x \in M$ be a nilpotent element, Δ the type determined by (W, g, x) and $X = X(\Delta) \in \Pi$ the corresponding chromosome. Similarly, let Δ_{σ} , resp. Δ_{τ} , be the type determined by (W', g_{σ}, x) , resp. (W', g_{τ}, x) , and let X_{σ} , resp. X_{τ} , be its chromosome in (Π, Λ) , resp. (Λ, Π) .

LEMMA 8. *With the above notation we have $X_{\sigma} = \sigma(X)$ and $X_{\tau} = \tau(X)$.*

PROOF. It suffices to prove this when Δ is indecomposable, i.e., X is a gene, say $X = g^{\epsilon}(m+1)$. We may assume (see §5) that there is a basis e_0, \dots, e_m of W such that

$$(11.1) \quad g(e_r, e_s) = \epsilon \delta_{r+s, m}$$

for all r, s and

$$(11.2) \quad xe_r = e_{r+1}i \quad (0 \leq r < m), \quad xe_m = 0.$$

Since W' is indecomposable with respect to x , we must have $X_{\sigma} = g(m+1)$ for m even and $X_{\tau} = g(m+1)$ for m odd. If m is odd we have $i^{m-1}\epsilon g_{\sigma}(e_0, x^m e_0) = -i^m \epsilon g(e_0, e_m i^m) = 1$ by (11.1), (11.2) and the definition of g_{σ} . By (5.5) we have

$X_\sigma = g^\varepsilon(m+1)$ in this case. If m is even then $i^m \varepsilon g_\tau(e_0, x^m e_0) = i^m \varepsilon g(e_0, e_m i^m) = 1$, and so by (5.4) we have $X_\tau = g^\varepsilon(m+1)$.

Thus $X_\sigma = \sigma(X)$ and $X_\tau = \tau(X)$ in all cases.

We now proceed with our proof of the sufficiency part of Theorem 5. We shall consider in detail the case $j = 7$. Then, by hypothesis, (9.1) is a primitive (Π, Λ) -mutation. By Lemma 7 there exists a primitive Π -mutation $Y_1 \rightarrow Y_2$ such that $\sigma(Y_r) = X_r$ ($r = 1, 2$).

From §5 it is easy to deduce that there exists a \mathbb{C} -subspace V_0 of V such that the canonical map $V_0 \otimes_{\mathbb{C}} \mathbf{H} \rightarrow V$ is an isomorphism, f takes complex values on $V_0 \times V_0$, and $\text{sig}(if_0) = \text{sig } Y_1 = \text{sig } Y_2$ where f_0 is the restriction of f to V_0 . Let U be the classical group of the fourth series defined by the pair (V_0, if_0) . Then we have $U_\sigma = G$. Let \mathcal{O}'_r ($r = 1, 2$) be the nilpotent orbit of U in its Lie algebra M with the associated chromosome Y_r . If we identify M_σ with L , then Lemma 8 implies that $\mathcal{O}'_r \subset \mathcal{O}_r$ ($r = 1, 2$). Since $Y_1 \leq Y_2$, we have $\mathcal{O}'_1 \leq \mathcal{O}'_2$ by the case $j = 4$. This clearly implies that $\mathcal{O}_1 \leq \mathcal{O}_2$.

The proof in the case $j = 10$ is similar and we may safely omit the details.

12. The cases $j = 6, 9$. These two cases will be treated simultaneously. Recall that $\Phi_6 = (2\Lambda, \Pi)$ and $\Phi_9 = (\Pi, 2\Lambda)$.

If $m (\geq 0)$ is an integer we shall write $m' = [\frac{1}{2}(m+1)]$. Thus $m = 2m'$ for m even and $m = 2m' - 1$ for m odd. We have

$$(12.1) \quad (-1)^{m-m'} = \begin{cases} i^m & \text{if } m \text{ is even,} \\ i^{m-1} & \text{if } m \text{ is odd.} \end{cases}$$

Assume first that the mutation (9.1) has the form

$$(12.2) \quad 2g(m+1) \rightarrow g^\varepsilon(m) + g^{-\varepsilon}(m+2).$$

We can choose a basis $e_0, \dots, e_m, e'_0, \dots, e'_m$ of V such that $f(e_r, e_s) = f(e'_r, e'_s) = 0$ and $f(e_r, e'_s) = \delta_{r+s, m}$ for all r, s . We define the one-parameter family $x(t) = x_0 + tx_1$ of operators in V by

$$\begin{aligned} x(t)e_0 &= e_1 - \varepsilon(-1)^{m+m'}te'_0 & \text{if } m > 0, \\ x(t)e_r &= e_{r+1} & (0 < r < m), \quad x(t)e_m = -\varepsilon(-1)^{m'}te'_m, \\ x(t)e'_r &= -e'_{r+1} & (0 \leq r < m), \quad x(t)e'_m = 0. \end{aligned}$$

It is easy to verify that $x(t) \in L$ for all real t . Since x_0 consists of two Jordan blocks of size $m+1$ each, we have $x_0 \in \mathcal{O}_1$. (Note that m is odd if $j = 6$ and m is even if $j = 9$.)

We have

$$(12.3) \quad x(t)^{m+1}e_0 = -2\varepsilon(-1)^{m'}te'_m \quad \text{and} \quad -\varepsilon i^{2m'}f(e_0, x(t)^{m+1}e_0) = 2t.$$

Now let $t > 0$. If $m = 0$ then $x(t)$ is a single Jordan block and by (5.4)–(5.5) we have $x(t) \in \mathcal{O}_2$. Now let $m \geq 1$. Then $x(t)^{m+2} = 0$ and $\dim \text{Ker } x(t) = 2$. It follows that $x(t)$ consists of two Jordan blocks whose sizes are $m+2$ and m . The larger block is the type $\Delta_{m+1}^{-\varepsilon}(0)$ because of (12.3). To avoid tedious computations which are needed to determine the type of the smaller block, we shall use the following ad

hoc argument. The smaller block is either of type $\Delta_{m-1}^\varepsilon(0)$ or of type $\Delta_{m-1}^{-\varepsilon}(0)$. When we let $t \rightarrow 0$ ($t > 0$) the type must become $\Delta_{m-1}^\varepsilon(0)$. Otherwise there is a strictly decreasing sequence t_1, t_2, \dots with limit 0 for which the smaller block is of type $\Delta_{m-1}^{-\varepsilon}(0)$. By the necessity part of Theorem 5, we then obtain $2g(m+1) \leq g^{-\varepsilon}(m+2) + g^{-\varepsilon}(m)$. By applying the priming operation $m-1$ times (see §6) we obtain $2g(2) \leq g^{-\varepsilon}(3) + g^{-\varepsilon}(1)$, which is false. Hence for small positive t the smaller block is of type $\Delta_{m-1}^\varepsilon(0)$. Consequently for small positive t we have $x(t) \in \mathcal{O}_2$. Since the set of all t for which the smaller block is of type $\Delta_{m-1}^\varepsilon(0)$ (or $\Delta_{m-1}^{-\varepsilon}(0)$) is open, it follows that in fact $x(t) \in \mathcal{O}_2$ for all $t > 0$.

Next assume that (9.1) has the form

$$(12.4) \quad g^\varepsilon(m+1) + g^{\varepsilon'}(n+1) \rightarrow g^\varepsilon(m-1) + g^{\varepsilon'}(n+3), \quad m \leq n.$$

We can choose a basis $e_0, \dots, e_m, e'_0, \dots, e'_n$ of V such that $f(e_r, e'_s) = 0$ for all r, s and $f(e_r, e_s) = \varepsilon \delta_{r+s, m}$ and $f(e'_r, e'_s) = \varepsilon' \delta_{r+s, n}$ for $r \leq s$. The family $x(t)$ is defined by

$$\begin{aligned} x(t)e_0 &= e_1 + \varepsilon t e'_0, \\ x(t)e_r &= e_{r+1} \quad (0 < r < m'), \\ x(t)e_r &= -e_{r+1} \quad (m' \leq r < m), \\ x(t)e_m &= 0, \\ x(t)e'_r &= e'_{r+1} \quad (0 \leq r < n'), \\ x(t)e'_r &= -e'_{r+1} \quad (n' \leq r < n), \\ x(t)e'_n &= -\varepsilon' t e_m. \end{aligned}$$

One can easily verify that x_0 and x_1 are in L and so $x(t) \in L$ for all real t . Moreover it is clear that $x_0 \in \mathcal{O}_1$ (see §5). We have

$$(12.5) \quad \begin{aligned} x(t)^{n+2}e_0 &= -\varepsilon \varepsilon' (-1)^{n-n'} t^2 e_m, \quad \text{and} \\ \varepsilon' (-1)^{n-n'+1} f(e_0, x(t)^{n+2}e_0) &= t^2. \end{aligned}$$

Now let $t > 0$. If $m = 1$ then $x(t)$ is a single Jordan block and (12.5) and (5.4)–(5.5) show that it is of type $\Delta_{n+2}^{\varepsilon'}(0)$. Thus $x(t) \in \mathcal{O}_2$ in this case. Now let $m \geq 2$. Since $x(t)^{n+3} = 0$ and $\dim \text{Ker } x(t) = 2$, $x(t)$ consists of two Jordan blocks whose sizes are $n+3$ and $m-1$. The larger block is of type $\Delta_{n+2}^{\varepsilon'}(0)$, by (12.5) and (5.4)–(5.5). As in the case (12.2) we can conclude that for $t > 0$ the other Jordan block of $x(t)$ must belong to the type $\Delta_{m-2}^\varepsilon(0)$. Thus $x(t) \in \mathcal{O}_2$ for $t > 0$.

Now assume that the mutation (9.1) has the form

$$(12.6) \quad \begin{aligned} g^\varepsilon(m+1) + g^+(n+1) + g^-(n+1) \\ \rightarrow g^\varepsilon(m-1) + 2g(n+2), \quad m \leq n. \end{aligned}$$

We can choose a basis $e_0, \dots, e_n, e'_0, \dots, e'_n, e''_0, \dots, e''_m$ of V such that $f(e_r, e_s) = f(e'_r, e'_s) = 0$, $f(e_r, e''_s) = f(e'_r, e''_s) = 0$, $f(e_r, e'_s) = \delta_{r+s, n}$ for all r, s and $f(e''_r, e''_s) = \varepsilon \delta_{r+s, m}$ for $r \leq s$. The family $x(t)$ of operators in V is defined by

$$\begin{aligned} x(t)e_r &= e_{r+1} \quad (0 \leq r < n), \quad x(t)e_n = 0, \\ x(t)e'_r &= -e'_{r+1} \quad (0 \leq r < n), \quad x(t)e'_n = -\varepsilon t e''_m, \end{aligned}$$

$$\begin{aligned} x(t)e_0'' &= e_1'' + te_0, \\ x(t)e_r'' &= e_{r+1}'' \quad (0 < r < m'), \\ x(t)e_r'' &= -e_{r+1}'' \quad (m' \leq r < m), \quad x(t)e_m'' = 0. \end{aligned}$$

One can check that $x(t) \in L$ for all real t . Let W, W' and W'' be the subspaces of V spanned by the vectors e_r, e_r' and e_r'' , respectively. These subspaces are invariant under $x_0 = x(0)$ and $W'' \perp (W + W')$. The restriction of x_0 to each of these subspaces is just a single Jordan block. The block in W'' is of type $\Delta_m^\varepsilon(0)$. The other two subspaces, W and W' , are totally isotropic. This implies that the restriction of x_0 to $W + W'$ belongs to the type $\Delta_n^+(0) + \Delta_n^-(0)$. Hence $x_0 \in \mathcal{O}_1$.

Now let $t \neq 0$. If $m = 1$ one can easily check that $x(t)$ consists of two Jordan blocks of size $n + 2$ and so $x(t) \in \mathcal{O}_2$. Now let $m \geq 2$. Let V_1 be the $x(t)$ -invariant subspace generated by e_0' . A basis of V_1 consists of the vectors e_r' ($0 \leq r \leq n$) and e_m'' . Let V_2 be the $x(t)$ -invariant subspace generated by e_0'' . A basis of V_2 consists of the vectors $e_0'', e_r'' + te_{r-1}$ ($1 \leq r \leq m'$), $(-1)^{r-m'}e_r'' + te_{r-1}$ ($m' < r \leq m$), e_r ($m \leq r \leq n$). We have $V_1 \cap V_2 = 0$ and $\dim V_1 = \dim V_2 = n + 2$. Since $x(t)^{n+2} = 0$ and

$$x(t)^{m-2}e_1'' = (-1)^{m-m'-1}e_m' \notin V_1 + V_2, \quad x(t)^{m-1}e_1'' \in V_1,$$

it follows that $x(t)$ consists of three Jordan blocks whose sizes are $n + 2, n + 2$, and $m - 1$. Since $\varepsilon(-1)^{m-m'-1}f(e_1', x(t)^{m-2}e_1'') = 1$, it follows that the Jordan block of size $m - 1$ belongs to the type $\Delta_{m-2}^\varepsilon(0)$. Hence $x(t) \in \mathcal{O}_2$ for $t \neq 0$.

Next assume that (9.1) has the form

$$(12.7) \quad g^+(m+1) + g^-(m+1) + g^\varepsilon(n+1) \rightarrow 2g(m) + g^\varepsilon(n+3), \quad m \leq n.$$

We can choose a basis $e_0, \dots, e_m, e_0', \dots, e_m', e_0'', \dots, e_n''$ of V such that $f(e_r, e_s) = f(e_r', e_s') = 0, f(e_r, e_s'') = f(e_r', e_s'') = 0, f(e_r, e_s') = \delta_{r+s, m'}$ for all r, s and $f(e_r', e_s'') = \varepsilon \delta_{r+s, n}$ for $r \leq s$. We define the family $x(t)$ by

$$\begin{aligned} x(t)e_0 &= e_1 - \varepsilon t e_0'' \quad (e_1 = 0 \text{ if } m = 0), \\ x(t)e_r &= e_{r+1} \quad (0 < r < m), \\ x(t)e_m &= 0 \quad \text{if } m > 0, \\ x(t)e_r' &= -e_{r+1}' \quad (0 \leq r < m), \quad x(t)e_m' = 0, \\ x(t)e_r'' &= e_{r+1}'' \quad (0 \leq r < n'), \\ x(t)e_r'' &= -e_{r+1}'' \quad (n' \leq r < n), \quad x(t)e_n'' = te_n'. \end{aligned}$$

One can check that $x(t) \in L$ for all real t . As in the case (12.6) one can show that $x_0 \in \mathcal{O}_1$.

Now let $t \neq 0$. An easy computation gives

$$(12.8) \quad \begin{aligned} x(t)^{n+2}e_0 &= (-1)^{n-n'+1}\varepsilon t^2 e_m, \quad \text{and} \\ \varepsilon(-1)^{n-n'+1}f(e_0, x(t)^{n+2}e_0) &= t^2. \end{aligned}$$

If $m = 0$ then (12.8) implies that $x(t) \in \mathcal{O}_2$. Now let $m \geq 1$. Then it is easy to verify that $x(t)$ consists of three Jordan blocks whose sizes are $n + 3, m$, and m . The

largest block corresponds to the $x(t)$ -invariant subspace generated by e_0 . It follows from (12.8) that this block belongs to the type $\Delta_{m+2}^\varepsilon(0)$, and so $x(t) \in \mathcal{O}_2$ for $t \neq 0$.

Next assume that the mutation (9.1) has the form

$$(12.9) \quad \begin{aligned} g^+(m+1) + g^-(m+1) + g^+(n+1) + g^-(n+1) \\ \rightarrow 2g(m) + 2g(n+2), \quad m \leq n. \end{aligned}$$

We can choose a basis $a_0, \dots, a_m, a'_0, \dots, a'_m, b_0, \dots, b_n, b'_0, \dots, b'_n$ of V such that $f(a_r, a_s) = f(a'_r, a'_s) = 0$, $f(b_r, b_s) = f(b'_r, b'_s) = 0$, $f(a_r, b_s) = f(a'_r, b_s) = f(a_r, b'_s) = f(a'_r, b'_s) = 0$, $f(a_r, a'_s) = \delta_{r+s, m}$, and $f(b_r, b'_s) = \delta_{r+s, n}$ for all r, s . We define the family $x(t)$ of operators in V by

$$\begin{aligned} x(t)a_0 &= a_1 + tb_0 \quad (a_1 = 0 \text{ if } m = 0), \\ x(t)a_r &= a_{r+1} \quad (0 < r < m), \\ x(t)a_m &= 0 \quad \text{if } m > 0, \\ x(t)a'_r &= -a'_{r+1} \quad (0 \leq r < m), \quad x(t)a'_m = 0, \\ x(t)b_r &= b_{r+1} \quad (0 \leq r < n), \quad x(t)b_n = 0, \\ x(t)b'_r &= -b'_{r+1} \quad (0 \leq r < n), \quad x(t)b'_n = -ta'_m. \end{aligned}$$

One can check that $x(t) \in L$ for all real t . By using the same argument as in the case (12.6) one can show that $x_0 \in \mathcal{O}_1$.

Now let $t \neq 0$. Let V_1 , resp. V_2 , be the $x(t)$ -invariant subspace generated by the vector a_0 , resp. b'_0 . We have

$$x(t)^{n+1}b'_0 = (-1)^n x(t)b'_n = (-1)^{n+1}ta'_m, \quad x(t)^{n+2}b'_0 = 0.$$

For $m > 0$ we have $x(t)^{n+1}a_0 = x(t)^n(a_1 + tb_0) = tx(t)^nb_0 = tb_n$, and, for $m = 0$, $x(t)^{n+1}a_0 = tx(t)^nb_0 = tb_n$. Thus for all m we have $x(t)^{n+1}a_0 = tb_n$ and $x(t)^{n+2}a_0 = 0$. It follows that $V_1 \cap V_2 = 0$. If $m = 0$ then $V = V_1 \oplus V_2$, and so $x(t) \in \mathcal{O}_2$. Now let $m \geq 1$. Then $x(t)^{n+2} = 0$ and $\dim \text{Ker } x(t) = 4$. Thus $x(t)$ consists of four Jordan blocks; the two largest of them arise from V_1 and V_2 and have size $n+2$. Since $x(t)^m V \subset V_1 + V_2$, the other two Jordan blocks must be of size m each. Therefore $x(t) \in \mathcal{O}_2$ for $t \neq 0$.

Next assume that the mutation (9.1) is of the form

$$(12.10) \quad g^\varepsilon(m+1) + g^{-\varepsilon}(n+1) \rightarrow g^\varepsilon(m-1) + g^\varepsilon(n+3), \quad m \leq n.$$

We can choose a basis $e_0, \dots, e_{m+k}, e'_0, \dots, e'_{m+k}$ ($k = n' - m'$) of V such that $f(e_r, e_s) = f(e'_r, e'_s) = 0$ and $f(e_r, e'_s) = \varepsilon(-1)^{n-m'+1}\delta_{r+s, m+k}$ for all r, s . We define the family $x(t)$ of operators in V by

$$\begin{aligned} x(t)e_r &= e_{r+1} \quad \text{for } r \neq k-1, k+1, k+m, \\ x(t)e_{k-1} &= e_k + (-1)^m e'_0 \quad \text{if } k > 0, \\ x(t)e_{k+1} &= e_{k+2} + (-1)^m t e'_0 \quad (e_{k+2} = 0 \text{ if } m = 1), \\ x(t)e_{k+m} &= -te'_{m-1} - e'_{m+1} \quad (e'_{m+1} = 0 \text{ if } k = 0), \\ x(t)e'_r &= -e'_{r+1} \quad (0 \leq r < k+m), \quad x(t)e'_{k+m} = 0. \end{aligned}$$

One can check that $x(t) \in L$ for all real t . If $k = 0$ then x_0 consists of two Jordan blocks of size $m+1$. Since the subspaces spanned by e_0, \dots, e_{m+k} and e'_0, \dots, e'_{m+k}

are x_0 -invariant and totally isotropic, we have $x_0 \in \Theta_1$. Now let $k \geq 1$. We have

$$\begin{aligned} x_0^n e_0 &= 2(-1)^k e'_{k+m}, & x_0^m ((-1)^m e'_0 - e_k) &= e'_m - e_{k+m}, \\ x_0^{n+1} e_0 &= 0, & x_0^{m+1} ((-1)^m e'_0 - e_k) &= 0. \end{aligned}$$

Hence x_0 consists of two Jordan blocks whose sizes are $n+1$ and $m+1$. Since

$$-\epsilon(-1)^{n-n'} f(e_0, x_0^n e_0) = 2, \quad \epsilon(-1)^{m-m'} f((-1)^m e'_0 - e_k, e'_m - e_{k+m}) = 2,$$

we conclude that again $x_0 \in \Theta_1$.

Now let $t > 0$. A simple computation gives

$$x(t)^{n+2} e_0 = \begin{cases} 2(-1)^k t e'_{k+m} & \text{if } m > 1, \\ (-1)^k t e'_{k+m} & \text{if } m = 1, \end{cases}$$

and so

$$(12.11) \quad \epsilon(-1)^{n-n'+1} f(e_0, x(t)^{n+2} e_0) > 0.$$

Thus if $m = 1$ then $x(t) \in \Theta_2$ for $t > 0$. Now let $m \geq 2$. Then $x(t)$ consists of two Jordan blocks of sizes $n+3$ and $m-1$. It follows from (12.11) that the larger block belongs to the type $\Delta_{n+2}^\epsilon(0)$. The same argument as in (12.2) implies that the other block belongs to the type $\Delta_{m-2}^{-\epsilon}(0)$. Therefore $x(t) \in \Theta_2$ for $t > 0$.

Next assume that the mutation (9.1) is of the form

$$(12.12) \quad 2g^\epsilon(m+1) + g^{-\epsilon}(n+1) \rightarrow 2g(m) + g^\epsilon(n+3), \quad m < n.$$

We can choose a basis $e_0, \dots, e_{m+k}, e'_0, \dots, e'_{m+k}, e''_0, \dots, e''_m$ ($k = n' - m'$) of V such that $f(e_r, e_s) = f(e'_r, e'_s) = 0$, $f(e_r, e''_s) = f(e'_r, e''_s) = 0$, $f(e_r, e'_s) = \epsilon \delta_{r+s, k+m}$ for all r, s and $f(e''_r, e''_s) = \epsilon \delta_{r+s, m}$ for $r \leq s$. The family of operators $x(t)$ in V is defined by

$$\begin{aligned} x(t)e_r &= e_{r+1} \quad \text{for } r \neq k-1, k+m-m', k+m, \\ x(t)e_{k-1} &= e_k - (-1)^{m'} e'_0, \\ x(t)e_{k+m-m'} &= e_{k+m-m'+1} + t e''_{m-m'} \quad \text{if } m > 0, \\ x(t)e_{k+m} &= (-1)^{m+m'} e'_{m+1} \quad \text{if } m > 0, \\ x(t)e_k &= (-1)^{m+m'} e'_{m+1} + t e''_{m-m'} \quad \text{if } m = 0, \\ x(t)e'_r &= -e'_{r+1} \quad (0 \leq r < k+m), \quad x(t)e'_{k+m} = 0, \\ x(t)e''_r &= e''_{r+1} \quad (0 \leq r < m'), \\ x(t)e''_{m'} &= -t e'_{m'} - e''_{m'+1} \quad (e''_{m'+1} = 0 \text{ if } m = m'), \\ x(t)e''_r &= -e''_{r+1} \quad (m' < r < m), \quad x(t)e''_m = 0 \quad \text{if } m > 1. \end{aligned}$$

One can check that $x(t) \in L$ for all real t . We have

$$(12.13) \quad \begin{aligned} x_0^n e_0 &= -2(-1)^{m+n'} e'_{k+m}, & x_0 e'_{k+m} &= 0, \\ -\epsilon(-1)^{n-n'} f(e_0, x_0^n e_0) &= 2. \end{aligned}$$

If $a = e_k + (-1)^{m'} e'_0$ and $b = e_{k+m} + (-1)^{m+m'} e'_m$ then we find that

$$(12.14) \quad x_0^m a = b, \quad x_0 b = 0, \quad \epsilon(-1)^{m-m'} f(a, x_0^m a) = 2.$$

Note also that the restriction of x_0 to the subspace spanned by the vectors e_r'' is a Jordan block of type $\Delta_m^\varepsilon(0)$. Hence it follows from (12.13) and (12.14) that $x_0 \in \mathcal{O}_1$.

Now let $t \neq 0$. An easy computation gives

$$(12.15) \quad x(t)^{n+2}e_0 = -(-1)^{k+m-m'}t^2e'_{k+m}, \quad \varepsilon(-1)^{n-n'+1}f(e_0, x(t)^{n+2}e_0) = t^2.$$

Thus if $m = 0$ then $x(t) \in \mathcal{O}_2$. Now let $m \geq 1$. Then $x(t)^{n+3} = 0$ and $\dim \text{Ker } x(t) = 3$. Thus $x(t)$ consists of three Jordan blocks. By (12.15) the largest block is of size $n+3$ and belongs to the type $\Delta_{n+2}^\varepsilon(0)$. Let W be the $x(t)$ -invariant subspace generated by e_0 . Then $\dim W = n+3$ and $x(t)^m V \subset W$. This implies that the remaining two Jordan blocks are of size m each. Therefore $x(t) \in \mathcal{O}_2$ also when $m \geq 1$.

Now assume that the mutation (9.1) is of the form

$$(12.16) \quad g^\varepsilon(m+1) + 2g^{-\varepsilon}(n+1) \rightarrow g^{-\varepsilon}(m-1) + 2g(n+2), \quad m < n.$$

We can choose a basis $e_0, \dots, e_{m+k}, e'_0, \dots, e'_{m+k}, e''_0, \dots, e''_n$ ($k = n' - m'$) of V such that $f(e_r, e_s) = f(e'_r, e'_s) = 0$, $f(e_r, e'_s) = f(e'_r, e''_s) = 0$, $f(e_r, e''_s) = -\varepsilon\delta_{r+s, m+k}$ for all r, s and $f(e'_r, e''_s) = -\varepsilon\delta_{r+s, n}$ for $r \leq s$. The family of operators $x(t)$ in V is defined by

$$\begin{aligned} x(t)e_0 &= e_1 + 2te''_0 \quad \text{if } k > 1, \\ x(t)e_0 &= e_1 - e'_0 + 2te''_0 \quad \text{if } k = 1, \\ x(t)e_r &= e_{r+1} \quad \text{for } r \neq 0, k-1, k+1, k+m, \\ x(t)e_{k-1} &= e_k + (-1)^{m'}e'_0 \quad \text{if } k > 1, \\ x(t)e_{k+1} &= e_{k+2} + 2t^2(-1)^{m'}e'_0 \quad \text{if } m > 1, \\ x(t)e_{k+m} &= -2t^2(-1)^{m+m'}e'_{m-1} - (-1)^{m+m'}e'_{m+1} \quad \text{if } m > 1, \\ x(t)e_{k+m} &= -4t^2e'_0 - e'_2 \quad \text{if } m = 1, \\ x(t)e'_r &= -e'_{r+1} \quad (0 \leq r < k+m), \quad x(t)e'_{k+m} = 0, \\ x(t)e''_r &= e''_{r+1} \quad (0 \leq r < n'), \\ x(t)e''_r &= -e''_{r+1} \quad (n' \leq r < n), \quad x(t)e''_n = -2te'_{k+m}. \end{aligned}$$

Note that $x(t)$ depends quadratically on t , i.e., $x(t) = x_0 + x_1t + x_2t^2$ where x_0, x_1, x_2 are constant operators and $x_2 \neq 0$. One can check that $x(t) \in L$ for all real t .

We have $x_0^n e_0 = 2(-1)^{m+n'}e'_{k+m}$, $x_0^{n+1}e_0 = 0$, and

$$(12.17) \quad -\varepsilon(-1)^{n-n'}f(e_0, x_0^n e_0) = 2.$$

If $a = e_k - (-1)^{m'}e'_0$ and $b = e_{k+m} - (-1)^{m+m'}e'_m$ then $x_0^m a = b$, $x_0 b = 0$, and

$$(12.18) \quad \varepsilon(-1)^{m-m'}f(a, x_0^m a) = 2.$$

Note also that the restriction of x_0 to the subspace spanned by the vectors e_r'' is a Jordan block of type $\Delta_n^{-\varepsilon}(0)$. Hence it follows from (12.17) and (12.18) that $x_0 \in \mathcal{O}_1$.

Now let $t \neq 0$. We have

$$x(t)^{n+2} = 0 \quad \text{and} \quad x(t)^{n+1}e_0 = 2t(-1)^{m+n'}(-2te'_{m+k-1} + e''_n).$$

Hence $x(t)$ has a Jordan block of size $n + 2$. But we know that such blocks must occur in pairs, and so there are two blocks of size $n + 2$. Thus if $m = 1$ then $x(t) \in \mathcal{O}_2$. Now let $m \geq 2$. Since $\dim \text{Ker } x(t) = 3$, $x(t)$ consists of three Jordan blocks. Hence the remaining Jordan block has size $m - 1$. By invoking the necessity part of Theorem 5 we can conclude that this block must be of type $\Delta_{m-2}^{-\epsilon}(0)$. Therefore $x(t) \in \mathcal{O}_2$.

Finally, assume that the mutation (9.1) has the form

$$(12.19) \quad 2g^{\epsilon}(m+1) + 2g^{-\epsilon}(n+1) \rightarrow 2g(m) + 2g(n+2), \quad m \leq n.$$

Let $V = V_1 \oplus V_2$ be an orthogonal decomposition with $V_1 = W_1 \oplus W'_1$, $V_2 = W_2 \oplus W'_2$ where W_1, W'_1, W_2, W'_2 are totally isotropic subspaces of dimension $m + k + 1$ each ($k = n' - m'$). We can choose bases a_0, \dots, a_{m+k} ; a'_0, \dots, a'_{m+k} ; b_0, \dots, b_{m+k} ; and b'_0, \dots, b'_{m+k} of W_1, W'_1, W_2 , and W'_2 , respectively, such that $f(a_r, a'_s) = f(b_r, b'_s) = (-1)^{n-m'+1} \epsilon \delta_{r+s, m+k}$ for all r, s . Let x_0 be the operator in V defined by

$$\begin{aligned} x_0 a_r &= a_{r+1} \quad \text{for } r \neq k-1, k+m, \\ x_0^0 a_{k-1} &= a_k + (-1)^m a'_0 \quad \text{if } k > 0, \\ x_0 a_{k+m} &= -a'_{m+1} \quad (a'_{m+1} = 0 \text{ if } k = 0), \\ x_0 a'_r &= -a'_{r+1} \quad (0 \leq r < m+k), \quad x_0 a'_{m+k} = 0, \end{aligned}$$

and by analogous formulas obtained from these by replacing a with b everywhere. From the case (12.10) it follows that $x_0 \in \mathcal{O}_1$. We make V into a complex vector space by defining the action of i by $ia_r = b_r$ and $ia'_r = b'_r$ for all r . We shall denote V with this complex structure by V' . Note that $a_0, \dots, a_{m+k}, a'_0, \dots, a'_{m+k}$ is a basis of V' . There exists a unique hermitian form f' on V' such that $\text{Re } f' = f$ for $j = 6$ (then f is symmetric) and $\text{Im } f' = f$ for $j = 9$ (then f is skew-symmetric). Indeed f' is determined by

$$\begin{aligned} f'(a_r, a'_s) &= f'(a'_r, a_s) = 0, \\ f'(a_r, a'_s) &= (-1)^{n-m'+1} \epsilon \delta_{r+s, m+k} \quad \text{if } j = 6, \\ f'(a_r, a'_s) &= (-1)^{n-m'+1} i \epsilon \delta_{r+s, m+k} \quad \text{if } j = 9. \end{aligned}$$

It is clear that f' is nondegenerate and $\text{sig}(f') = (k+m, k+m)$. Let U be the classical group of the fourth series defined by the pair (V', f') . Clearly U is a subgroup of G and we shall view its Lie algebra M as a subalgebra of L . Note that x_0 is a complex linear operator in V' and that $x_0 \in M$.

Let $Y_1 = g^{\epsilon}(m+1) + g^{-\epsilon}(n+1)$ and $Y_2 = g^{-\epsilon}(m) + g^{\epsilon}(n+2)$. Note that $\text{sig } Y_1 = \text{sig } Y_2 = (k+m, k+m)$. Let \mathcal{O}'_r ($r = 1, 2$) be the nilpotent orbit of U in M corresponding to the chromosome Y_r . Note that x_0 as a complex linear operator consists of two Jordan blocks whose sizes are $n+1$ and $m+1$.

We claim that $x_0 \in \mathcal{O}'_1$. If $n = m$ this follows from the fact that x_0 has a totally isotropic Jordan block, for instance, its restriction to the subspace of V' spanned by the vectors a_r . Now let $k > 0$. Then by using $x_0^n a_0 = 2(-1)^k a'_{m+k}$ and (12.1) we find that $(-i)^n (-\epsilon) f'(a_0, x_0^n a_0) = 2$, and so by (5.3) the larger block of x_0 is of type $\Delta_n^{-\epsilon}(0)$. Similarly one can check that the other block is of type $\Delta_m(0)$, and consequently

$x_0 \in \mathcal{O}'_1$. This implies that $\mathcal{O}'_1 \subset \mathcal{O}_1$. On the other hand it is clear that $\mathcal{O}'_2 \subset \mathcal{O}_2$. Since $Y_1 \leq Y_2$, we have $\mathcal{O}'_1 \leq \mathcal{O}'_2$ by the case $j = 4$ of Theorem 5. Consequently $\mathcal{O}_1 \leq \mathcal{O}_2$.

Since every primitive $(2\Lambda, \Pi)$ or $(\Pi, 2\Lambda)$ -mutation has one of the forms (12.2), (12.3), (12.6), (12.7), (12.9), (12.10), (12.12), (12.16), or (12.19), the proof of Theorem 5 is completed.

13. Closures of conjugacy classes. So far we have studied the closures of the orbits of a classical group G in its Lie algebra L . It is not hard to transfer these results to the conjugacy classes of G . In this section we introduce the necessary concepts and state the main results. The proofs either follow from the case of orbits in L or are obvious modifications of proofs already given, and so they will be omitted.

Each $x \in G$ has a unique multiplicative Jordan decomposition $x = x_s x_u$, where x_s is semisimple, x_u is unipotent, both x_s and x_u are in G , and $x_s x_u = x_u x_s$. We refer to x_s (resp. x_u) as the *semisimple* (resp. *unipotent*) *component* of x .

Recall that G is a group of \mathbf{F} -linear transformations of a vector space V which may be equipped with a form f (see §1). We assume that it belongs to the j th series. For $x \in G$ and $\lambda \in \mathbf{C}^*$ let

$$V(\lambda, x) = \text{Ker}(x - \lambda)^\infty$$

if $\mathbf{F} = \mathbf{C}$. Otherwise let

$$V(\lambda, x) = \text{Ker}(x^2 - 2 \text{Re}(\lambda)x + |\lambda|^2)^\infty.$$

If $j \in \{5, 8\}$ and $\lambda \neq \pm 1$ or $j \in \{4, 6, 7, 9, 10\}$ and $|\lambda| \neq 1$ then $V(\lambda, x)$ is totally isotropic. This leads us to define the subspace $V'(\lambda, x)$ by

$$V'(\lambda, x) = \begin{cases} V(\lambda, x) & \text{for } j \in \{1, 2, 3\}, \\ V(\lambda, x) + V(\lambda^{-1}, x) & \text{for } j \in \{5, 8\}, \\ V(\lambda, x) + V(\bar{\lambda}^{-1}, x) & \text{for } j \in \{4, 6, 7, 9, 10\}. \end{cases}$$

If $4 \leq j \leq 10$ the subspaces $V'(\lambda, x)$ are nondegenerate and orthogonal to each other.

We obtain a direct (and orthogonal if $j \geq 4$) decomposition

$$(13.1) \quad V = \sum_{\lambda \in \Omega_j^*} V'(\lambda, x)$$

where Ω_j^* is a subset of \mathbf{C}^* specified by the Table V.

j	Ω_j^*
1	\mathbf{C}^*
2, 3	$\text{Im } \lambda \geq 0$
4	$ \lambda \geq 1$
5, 8	$\text{Im } \lambda > 0$ or $(\lambda \in \mathbf{R}^* \& \lambda \geq 1)$
6, 7, 9, 10	$\text{Im } \lambda \geq 0 \& \lambda \geq 1$

TABLE V

For $\lambda \in \mathbf{C}^*$ let x_λ (resp. x'_λ) be the restriction of x to $V(\lambda, x)$ (resp. $V'(\lambda, x)$). If $1 \leq j \leq 3$ let $G(\lambda, x)$ be the general linear group of the space $V(\lambda, x)$. If $4 \leq j \leq 10$ let $G(\lambda, x)$ be the classical group defined by the subspace $V'(\lambda, x)$ and the restriction of the form f to this subspace. Thus in all cases $G(\lambda, x)$ is also a classical group of the j th series. Clearly $x'_\lambda \in G(\lambda, x)$ for all λ . We denote by $C(\lambda, x)$ the centralizer of x'_λ in $G(\lambda, x)$.

With this notation we have

$$(13.2) \quad C_G(x) = \prod_{\lambda \in \Omega^*} C(\lambda, x).$$

The following analog of Theorem 1 is valid.

THEOREM 1'. *Let G be a classical group of the j th series and $x \in G$ a semisimple element. Then each $C(\lambda, x)$ in (13.2) is a classical group of k th series where k depends only on j and λ and is specified in the last column of Table III.*

The group G acts on itself by conjugation. We shall write $a \cdot x = axa^{-1}$ for the result of the action of the element a on $x \in G$. Then $G \cdot x = \{axa^{-1} : a \in G\}$ is the conjugacy class of x in G . The following analog of Theorem 2 is also valid.

THEOREM 2'. *Let G be a classical group, $x, y \in G$ and $x = x_s x_u$ the Jordan decomposition of x . Then $y \in \overline{G \cdot x}$ iff $a \cdot y = x_s z$ for some $a \in G$ and some*

$$(13.3) \quad z \in \overline{C_G(x_s) \cdot x_u}.$$

This theorem reduces the problem of closures of conjugacy classes to the case of unipotent conjugacy classes. For this reason we shall now restrict our attention only to unipotent conjugacy classes.

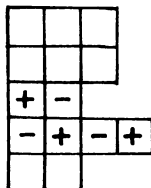
Let L be the Lie algebra of G and $\exp: L \rightarrow G$ the exponential map of G . It is known that \exp induces a G -equivariant homeomorphism of the set of nilpotent elements of L with the set of unipotent elements of G . (In particular, there is a bijection between the nilpotent orbits of G in L and the unipotent conjugacy classes of G .) This clearly solves the problem of closures of conjugacy classes of G .

14. Proof of Theorem 6: preliminaries. The remainder of this paper is devoted to the proof of Theorem 6. In this section we prove an important lemma (Lemma 9) and explain the method of proof.

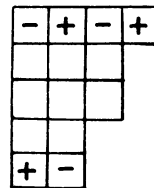
It is convenient to represent chromosomes by diagrams. For instance the chromosome

$$X = 2g(3) + g^-(2) + g(2) + g^+(4)$$

can be represented by the diagram



OR



The rows of such diagrams can be permuted arbitrarily (they correspond to the constituent genes of X). We shall usually arrange rows in decreasing rank as in the second diagram above.

The priming operation on chromosomes was defined in §6. It is clear that the diagram of X' can be obtained from the diagram of X by deleting its first column. The priming operation can be extended to varieties. Thus if Φ is a variety then $\Phi' = \{X': X \in \Phi\}$ is also a variety. By iterating, we obtain the varieties $\Phi^{(k)}$, $k \geq 0$, where $\Phi^{(0)} = \Phi$, $\Phi^{(1)} = \Phi'$, etc.

For instance we have $\Lambda' = \Lambda$, $\Pi' = \Pi$, $(2\Lambda, \Lambda)' = (\Lambda, 2\Lambda)$, $(\Lambda, 2\Lambda)' = (2\Lambda, \Lambda)$, $(\Lambda, \Pi)' = (\Pi, \Lambda)$, $(\Pi, \Lambda)' = (\Lambda, \Pi)$, $(2\Lambda, \Pi)' = (\Pi, 2\Lambda)$, and $(\Pi, 2\Lambda)' = (2\Lambda, \Pi)$.

The following lifting property of mutations will be often used in subsequent sections.

LEMMA 9 (LIFTING PROPERTY). *Let Φ be one of the varieties (6.2). If $X \in \Phi$ and $X^{(k)} \rightarrow U$ is a $\Phi^{(k)}$ -mutation, then there exists a Φ -mutation $X \rightarrow Z$ such that $Z^{(k)} = U$ and $\text{sig } X^{(i)} = \text{sig } Z^{(i)}$ for $0 \leq i \leq k$.*

PROOF. It suffices to consider the case when $X^{(k)} \rightarrow U$ is a primitive $\Phi^{(k)}$ -mutation and $X^{(k)}$ has the same number of genes (counting multiplicities) as X . The rest of the proof consists in routine verifications.

We shall give the details only in the case when $X^{(k)} \rightarrow U$ is given by the formula (8.17), i.e.,

$$X^{(k)} = g^\epsilon(m) + 2g^{-\epsilon}(n), \quad U = g^{-\epsilon}(m-2) + 2g(n+1),$$

where $1 < m \leq n$ and $\epsilon = \pm$. Note that $\Phi^{(k)} = (\Pi, 2\Lambda)$ if m and n are even and $\Phi^{(k)} = (2\Lambda, \Pi)$ when they are odd. (It is understood in (8.17) that $n-m$ is even.) Hence $\Phi = (\Pi, 2\Lambda)$ if $m-k$ is even and $\Phi = (2\Lambda, \Pi)$ if $m-k$ is odd. Since we assume that X and $X^{(k)}$ have the same number of genes, we must have $X = g^\epsilon(m+k) + 2g^{-\epsilon}(n+k)$. We claim that the Φ -mutation $X \rightarrow Z$ where

$$Z = g^{-\epsilon}(m+k-2) + 2g(n+k+1)$$

has the desired properties. Indeed it is easy to check that $\text{sig } X^{(i)} = \text{sig } Z^{(i)}$ for $0 \leq i \leq k$ and it is obvious that $Z^{(k)} = U$.

The Φ -mutation $X \rightarrow Z$ having the properties mentioned in Lemma 9 will be called a *lifting* of the $\Phi^{(k)}$ -mutation $X^{(k)} \rightarrow U$.

Let Φ be one of the varieties (6.2) and $X, Y \in \Phi$ such that $X < Y$ (i.e., $X \leq Y$ and $X \neq Y$) and $\text{sig } X = \text{sig } Y$. In particular, X and Y have the same rank, say n , and so $X, Y \in \Phi(n)$. In order to prove Theorem 6 it suffices to prove the following statement:

(*) There exists a Φ -mutation $X \rightarrow Z$ such that $Z \leq Y$.

The proof of this statement will be given by induction on the rank n . The verification of (*) for small values of n is straightforward and is omitted. The proof is divided into three sections: the next section treats the case $\Phi = \Pi$, the subsequent section the cases $\Phi = (\Pi, \Lambda)$ and (Λ, Π) , and the last one the cases $\Phi = (\Pi, 2\Lambda)$ and $(2\Lambda, \Pi)$.

15. The case $\Phi = \Pi$. Before starting the proof we introduce some notations. To each $X \in \Pi$ we associate a $2 \times \infty$ nonnegative integral matrix $\sigma(X)$ defined by

$$(15.1) \quad \sigma(X) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ b_0 & b_1 & b_2 & \cdots \end{pmatrix}$$

where $(a_k, b_k) = \text{sig } X^{(k)}$ for all k . It is easy to see that the entries of this matrix satisfy the following conditions:

$$(15.2) \quad a_0 \geq a_1 \geq a_2 \geq \dots, \quad a_k = 0 \quad \text{for large } k;$$

$$(15.3) \quad b_0 \geq b_1 \geq b_2 \geq \dots, \quad b_k = 0 \quad \text{for large } k;$$

$$(15.4) \quad a_0 \geq b_1 \geq a_2 \geq b_3 \geq \dots;$$

$$(15.5) \quad b_0 \geq a_1 \geq b_2 \geq a_3 \geq \dots;$$

$$(15.6) \quad a_0 - a_1 \geq b_1 - b_2 \geq a_2 - a_3 \geq b_3 - b_4 \geq \dots;$$

$$(15.7) \quad b_0 - b_1 \geq a_1 - a_2 \geq b_2 - b_3 \geq a_3 - a_4 \geq \dots$$

The diagram of X can be recovered from $\sigma(X)$ as follows: The first column of the diagram consists of $a_0 - a_1$ positive and $b_0 - b_1$ negative nodes; the second column consists of $a_1 - a_2$ positive and $b_1 - b_2$ negative nodes, etc. Hence the map σ is a bijection between Π and the set of all $2 \times \infty$ integral matrices whose entries satisfy the conditions (15.2)–(15.7).

Let $X, Y \in \Pi(n)$ satisfy $X < Y$. We shall show that there exists a Π -mutation $X \rightarrow Z$ such that $Z \leq Y$. We use induction on n . For $n = 0, 1$ the assertion is vacuously true; so let $n > 1$. Let $\sigma(X)$ be as above and let

$$\sigma(Y) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots \\ d_0 & d_1 & d_2 & \cdots \end{pmatrix}.$$

Since $X < Y$, we have

$$(15.8) \quad a_k \leq c_k, \quad b_k \leq d_k \quad (k \geq 0).$$

Since $a_0 + b_0 = n = c_0 + d_0$, it follows from (15.8) that $a_0 = c_0$ and $b_0 = d_0$.

Assume that X and Y are not disjoint, i.e., there is a gene g such that $X = X_1 + g$ and $Y = Y_1 + g$, with $X_1, Y_1 \in \Pi$. Then $X_1 < Y_1$ and by induction hypothesis there exists a Π -mutation $X_1 \rightarrow Z_1$ such that $Z_1 \leq Y_1$. Then $X \rightarrow Z = Z_1 + g$ is also a Π -mutation and $Z \leq Y$. Thus we may assume (and we do) that X and Y are disjoint.

Assume now that $(a_k, b_k) = (c_k, d_k) \neq (0, 0)$ for some $k \geq 1$. Since $X < Y$, we have $X^{(k)} \leq Y^{(k)}$. Since X and Y are disjoint, so are $X^{(k)}$ and $Y^{(k)}$ and consequently $X^{(k)} < Y^{(k)}$. By induction hypothesis there exists a Π -mutation $X^{(k)} \rightarrow U$ such that $U \leq Y^{(k)}$. By the lifting property (Lemma 9) there exists a Π -mutation $X \rightarrow Z$ such that $Z^{(k)} = U$ and $Z \leq Y$. Hence from now on we may also assume that

$$(15.9) \quad Y^{(k)} \neq 0 \text{ \& } k \geq 1 \Rightarrow a_k < c_k \text{ or } b_k < d_k.$$

Now assume that $X \supset g^+(k) + g^-(k)$ for some k . Then Y contains no genes of rank k . Since $X < Y$, we have $Y^{(k)} \neq 0$. By (15.9) we have $a_k < c_k$ or $b_k < d_k$. We define a Π -mutation $X \rightarrow Z$ by $g^+(k) + g^-(k) \rightarrow g^e(k+1) + g^{-e}(k-1)$, where

$\varepsilon = +$ if $a_k < c_k$ and $\varepsilon = -$ otherwise. Clearly $Z \leq Y$. Hence we may assume from now on (and we do) that

$$(15.10) \quad X \not\supset g^+(k) + g^-(k), \quad k \geq 1.$$

Until further notice we shall assume that $a_1 < c_1$. Let $g_1 = g_{\varepsilon_1}(m)$ be a gene of X with m minimal.

Case 1: $\varepsilon_1 = -$. Since $a_1 < c_1$, X contains a gene of the form $g_2 = g_+(k)$. Assume also that k is minimal. We define a Π -mutation $X \rightarrow Z$ by $g_1 + g_2 \rightarrow g_-(k+1) + g_+(m-1)$. If k is odd we have

$$\begin{aligned} c_{k-1} - c_k &\leq d_{k-2} - d_{k-1} \leq \dots \leq c_0 - c_1 < a_0 - a_1 \\ &= b_1 - b_2 = \dots = b_{k-2} - b_{k-1} = a_{k-1} - a_k. \end{aligned}$$

Therefore $a_k < c_k$, $b_{k-1} < d_{k-1}$, $a_{k-2} < c_{k-2}$, \dots , $b_2 < d_2$ and so $Z \leq Y$. If k is even we have $d_{k-1} - d_k \leq c_{k-2} - c_{k-1} \leq \dots \leq c_0 - c_1 < a_0 - a_1 = b_1 - b_2 = \dots = a_{k-2} - a_{k-1} = b_{k-1} - b_k$. Hence $b_k < d_k$, $a_{k-1} < c_{k-1}$, $b_{k-2} < d_{k-2}$, \dots , $b_2 < d_2$, and again we have $Z \leq Y$.

Case 2: $g_1 = g_+(1)$. Since $b_0 = c_0 \geq c_1 > a_1$, X must contain a negative gene. Let $g_2 = g^-(k)$ be such a gene with k minimal. Define a Π -mutation $X \rightarrow Z$ by $g_1 + g_2 \rightarrow g^+(k+1)$. For $1 \leq i \leq k$ we have

$$d_0 - d_{i-1} \geq c_1 - c_i > a_1 - c_i = (a_1 - a_i) + (a_i - c_i) = b_0 - b_{i-1} + (a_i - c_i).$$

Since $b_0 = d_0$ and $b_{i-1} \leq d_{i-1}$, we infer that $a_i < c_i$, and so $Z \leq Y$.

Case 3: $\varepsilon_1 = +$, $m \geq 2$, and $X \supset 2g_1$. We define the Π -mutation $X \rightarrow Z$ by $2g_1 \rightarrow g_+(m-2) + g_+(m+2)$. We have $b_1 - b_2 = a_0 - a_1 > c_0 - c_1 \geq d_1 - d_2$, and so $b_2 < d_2$. Let m be even and $i = m-1$ or m . Then $c_1 - c_i \leq d_0 - d_{i-1} \leq b_0 - b_{i-1} = a_1 - a_i$ and $d_2 - d_{i+1} \leq d_0 - d_{i-1} \leq b_0 - b_{i-1} = b_2 - b_{i+1}$. Consequently, $a_i < c_i$ and $b_{i+1} < d_{i+1}$, and so $Z \leq Y$. Now let m be odd and $i = m-1$ or m . Then $c_1 - c_{i+1} \leq d_0 - d_i \leq b_0 - b_i = a_1 - a_{i+1}$ and $d_2 - d_i \leq d_0 - d_{i-2} \leq b_0 - b_{i-2} = b_2 - b_i$. Since $a_1 < c_1$ and $b_2 < d_2$, it follows that $a_{i+1} < c_{i+1}$ and $b_i < d_i$. Thus we have again $Z \leq Y$.

Case 4: $\varepsilon_1 = +$, $m \geq 2$, and $X \not\supset 2g_1$. Clearly $X^{(m)} \neq 0$, and let $g_2 = g^{\varepsilon_2}(k)$ be a gene of X with k minimal subject to $k > m$. Note that $g_1 = g^\varepsilon(m)$ where $\varepsilon = (-1)^{m-1}$.

Let first $\varepsilon_2 = -\varepsilon$. Then we define a Π -mutation $X \rightarrow Z$ by $g_1 + g_2 \rightarrow g^\varepsilon(k+1) + g^{-\varepsilon}(m-1)$. As in Case 3 we have $a_m < c_m$ and $b_m < d_m$. Let m be even and $m \leq i \leq k-1$. For i even we have $d_i - d_{i+1} \leq d_0 - d_1 \leq b_0 - b_1 = b_i - b_{i+1}$, and for i odd

$$\begin{aligned} d_i - d_{i+1} &\leq d_1 - d_2 \leq c_0 - c_1 \leq a_0 - a_1 - 1 \\ &= b_{m-1} - b_m - 1 = a_m - a_{m+1} = b_i - b_{i+1}. \end{aligned}$$

Since $b_m < d_m$, we obtain $b_i < d_i$ ($m \leq i \leq k$), and so $Z \leq Y$. Now let m be odd and $m \leq i \leq k-1$. For i even we have

$$c_i - c_{i+1} \leq c_0 - c_1 \leq a_0 - a_1 - 1 = a_{m-1} - a_m - 1 = b_m - b_{m+1} = a_i - a_{i+1},$$

and for i odd $c_i - c_{i+1} \leq c_1 - c_2 \leq d_0 - d_1 \leq b_0 - b_1 = a_i - a_{i+1}$. Since $a_m < c_m$, we obtain $a_i < c_i$ ($m \leq i \leq k$), and so we have again $Z \leq Y$.

Now let $\varepsilon_2 = \varepsilon$ and let $k - m$ be even. Then we define a Π -mutation $X \rightarrow Z$ by $g_1 + g_2 \rightarrow g^\varepsilon(k + 2) + g^\varepsilon(m - 2)$. As in Case 3 we have $a_{m-1} < c_{m-1}$, $a_m < c_m$, $b_m < d_m$ and, if $m > 2$, $b_{m-1} < d_{m-1}$. Let m be even and $m \leq i \leq k - 1$. For i even we have

$$c_i - c_{i+1} \leq c_0 - c_1 \leq a_0 - a_1 - 1 = a_m - a_{m+1} = a_i - a_{i+1},$$

and for i odd $c_i - c_{i+1} \leq c_1 - c_2 \leq d_0 - d_1 \leq b_0 - b_1 = a_i - a_{i+1}$. It follows that $a_i < c_i$ for $m - 1 \leq i \leq k$. The proof of $b_i < d_i$ ($m \leq i \leq k + 1$) is the same as when $\varepsilon_2 = -\varepsilon$ except that one has to use also the equality $a_{k-1} - a_k = b_k - b_{k+1}$. Thus $Z \leq Y$. Now let m be odd and $m \leq i \leq k - 1$. For i even we have $d_i - d_{i+1} \leq d_0 - d_1 \leq b_0 - b_1 = b_i - b_{i+1}$, and for i odd

$$d_i - d_{i+1} \leq c_0 - c_1 \leq a_0 - a_1 - 1 = b_m - b_{m+1} = b_i - b_{i+1}.$$

Thus $b_i < d_i$ for $m - 1 \leq i \leq k$. The proof of $a_i < c_i$ ($m \leq i \leq k + 1$) is the same as when $\varepsilon_2 = -\varepsilon$ except that one has to use also the equality $b_{k-1} - b_k = a_k - a_{k+1}$. Hence again $Z \leq Y$.

Finally let $\varepsilon_2 = \varepsilon$ and let $k - m$ be odd. Then we define the Π -mutation $X \rightarrow Z$ by $g_1 + g_2 \rightarrow g^\varepsilon(k + 1) + g^\varepsilon(m - 1)$. The proof of $Z \leq Y$ is the same as when $k - m$ is even.

This completes the Case 4 and exhausts all possibilities when $a_1 < c_1$. If $b_1 < d_1$ then by interchanging positive and negative charges in both X and Y , we reduce this case to the case $a_1 < c_1$. In view of (15.9) this completes the proof.

16. The cases $\Phi = (\Pi, \Lambda)$ or (Λ, Π) . We shall treat the two cases simultaneously. All mutations in this proof are understood to be Φ -mutations. Let $X, Y \in \Phi(n)$ and $X < Y$. We shall show that there exists a mutation $X \rightarrow Z$ such that $Z \leq Y$. The proof is by induction on n . For $n = 0, 1$ the assertion is vacuously true, so let $n > 1$. We shall write $\text{sig } X^{(i)} = (a_i, b_i)$, $\text{sign } Y^{(i)} = (c_i, d_i)$, $r_i = a_i + b_i$, $s_i = c_i + d_i$. Since $X < Y$, we have $(a_i, b_i) \leq (c_i, d_i)$ for all i . Since $a_0 + b_0 = n = c_0 + d_0$, we have $a_0 = c_0$ and $b_0 = d_0$. If $\Phi = (\Lambda, \Pi)$ (resp., $\Phi = (\Pi, \Lambda)$) then $a_i = b_i$ and $c_i = d_i$ for i odd (resp., i even), and a_i, b_i, c_i, d_i are integers for i even (resp. i odd). As in the previous section we may assume that X and Y are disjoint. In view of the lifting property and our induction hypothesis, we may assume (and we do) that

$$(16.1) \quad Y^{(i)} \neq 0 \text{ \& } i \geq 1 \Rightarrow r_i < s_i.$$

Assume that $X \supset g^+(m) + g^-(m)$ for some m . We define the mutation $X \rightarrow Z$ by $g^+(m) + g^-(m) \rightarrow g(m - 1) + g(m + 1)$. Since X and Y are disjoint, we have $Y^{(m)} \neq 0$. Then (16.1) implies that $r_m < s_m$, and consequently $Z \leq Y$. Hence from now on we may assume that $X \not\supset g^+(m) + g^-(m)$, $m \geq 1$.

Next assume that $X \supset 2g(m)$ for some m . Since X and Y are disjoint and $X < Y$, we have $Y^{(m)} \neq 0$. By (16.1) we have $a_m < c_m$ or $b_m < d_m$. Then we have $X \rightarrow Z \leq Y$ where the mutation $X \rightarrow Z$ is defined by $2g(m) \rightarrow g^{-\varepsilon}(m - 1) + g^\varepsilon(m + 1)$; $\varepsilon = +$ if $a_m < c_m$ and $\varepsilon = -$ if $a_m < c_m$. Without loss of generality we assume from now on that $X \not\supset 2g(m)$, $m \geq 1$.

From now on let g_1 be a gene of X of minimum rank $r(g_1) = m$.

When g_1 is nonpolarized we denote by g_2 a gene of $X - g_1$ of minimum rank $r(g_2) = k$. Since $g_1 = g(m)$ and $X \not\supset 2g(m)$, we have $m < k$. Then we distinguish two cases.

Case 1: $g_2 = g(k)$. By (16.1) $r_m < s_m$, and so $a_m < c_m$ or $b_m < d_m$. By switching + and - charges in all polarized genes of X and Y (if necessary) we may assume that $a_m < c_m$. If $i - m$ is even and $m \leq i \leq k - 2$ then

$$c_i - c_{i+2} \leq s_i - s_{i+1} \leq s_0 - s_1 \leq r_0 - r_1 - 1 = r_i - r_{i+1} = a_i - a_{i+2}.$$

Since $a_m < c_m$, these inequalities imply that $a_j < c_j$ for $m \leq j \leq k$. Hence we have $X \rightarrow Z \leq Y$, where $X \rightarrow Z$ is defined by $g(m) + g(k) \rightarrow g^-(m-1) + g^+(k+1)$.

Case 2: g_2 is polarized. Without loss of generality we may assume that $g_2 = g^+(k)$. If $i - m$ is even and $m \leq i \leq k - 3$ then $d_i - d_{i+2} \leq s_i - s_{i+1} \leq s_0 - s_1 \leq r_0 - r_1 - 1 = r_i - r_{i+1} = b_i - b_{i+2}$. If $m \geq 2$ we have

$$d_{m-2} - d_m \leq s_{m-2} - s_{m-1} \leq s_0 - s_1 < r_0 - r_1 = r_{m-2} - r_{m-1} = b_{m-2} - b_{m-1},$$

and so $b_m < d_m$. Thus if $m \geq 2$ or if $m = 1$ and $b_1 < d_1$, it follows that $b_j < d_j$ for $m \leq j \leq k - 1$. Under these conditions we have $X \rightarrow Z \leq Y$, where $X \rightarrow Z$ is defined by $g(m) + g^+(k) \rightarrow g^+(m-1) + g(k+1)$. Now let $m = 1$ and $b_1 = d_1$. Then $c_1 - a_1 = s_1 - r_1 \geq 1$ and consequently $X - g_1 - g_2$ must contain either a negative or a nonpolarized gene. Let g_3 be such a gene of minimal rank $r(g_3) = t$. If i is even and $2 \leq i \leq t - 1$ then

$$c_i - c_{i+1} \leq c_0 - c_1 \leq a_0 - a_1 - 1 < b_1 - b_2 = a_2 - a_3 = \dots = a_i - a_{i+1}.$$

Hence $a_j < c_j$ for $1 \leq j \leq t$. If t is even then $g_3 = g^-(t)$ and we have $X \rightarrow Z \leq Y$, with $X \rightarrow Z$ defined by $g_2 + g_3 \rightarrow g(k-1) + g(t+1)$. If t is odd then $g_3 = g(t)$ and we have $X \rightarrow Z \leq Y$, with $X \rightarrow Z$ defined by $g_2 + g_3 \rightarrow g(k-1) + g^+(t+1)$.

Now assume that g_1 is polarized. Without any loss of generality we assume that $g_1 = g^+(m)$. We now distinguish several cases.

Case 3: $m \geq 3$. We have

$$\begin{aligned} c_{m-1} - c_{m+1} &\leq s_{m-1} - s_m \leq s_0 - s_1 < r_0 - r_1 = a_{m-1} - a_{m+1}, \\ d_{m-3} - d_{m-1} &\leq s_{m-3} - s_{m-2} \leq s_0 - s_1 < r_0 - r_1 = b_{m-3} - b_{m-1}, \end{aligned}$$

and so $a_{m+1} < c_{m+1}$ and $b_{m-1} < d_{m-1}$. If $X \supset 2g_1$ then we have $X \rightarrow Z \leq Y$, where $X \rightarrow Z$ is defined by $2g_1 \rightarrow g^+(m-2) + g^+(m+2)$. We may now assume that $X \not\supset 2g_1$, and let g_2 be a gene of $X - g_1$ of minimal rank $r(g_2) = k$. If $i - m$ is odd and $m + 1 \leq i \leq k - 2$, we have $c_i - c_{i+2} \leq s_i - s_{i+1} \leq s_0 - s_1 \leq r_0 - r_1 - 1 = r_i - r_{i+1} = a_i - a_{i+2}$. It follows that $a_j < c_j$ for $m \leq j \leq k$. If $g_2 = g^+(k)$ we also obtain by the same argument that $c_{k-1} - c_{k+1} \leq a_{k-1} - a_{k+1}$, and $a_{k+1} < c_{k+1}$. If $i - m$ is odd and $m - 1 \leq i \leq k - 2$, we have

$$d_i - d_{i+2} \leq s_i - s_{i+1} \leq s_0 - s_1 \leq r_0 - r_1 - 1 = r_{i+1} - r_{i+2} = b_i - b_{i+2}.$$

Since $b_{m-1} < d_{m-1}$, we infer that $b_j < d_j$ for $m - 1 \leq j \leq k$. Hence we have $X \rightarrow Z \leq Y$, where $X \rightarrow Z$ is defined by

$$\begin{aligned} g_1 + g_2 &\rightarrow g(m-1) + g^+(k+1) && \text{if } g_2 = g(k); \\ g_1 + g_2 &\rightarrow g(m-1) + g(k+1) && \text{if } g_2 = g^-(k); \\ g_1 + g_2 &\rightarrow g^+(m-2) + g^+(k+2) && \text{if } g_2 = g^+(k). \end{aligned}$$

Case 4: $m = 2$. Since $c_1 - c_3 \leq s_1 - s_2 \leq s_0 - s_1 < r_0 - r_1 = a_1 - a_3$ and $a_1 \leq c_1$, we have $a_3 < c_3$. If $b_1 < d_1$ then we can proceed exactly as in Case 1. Thus we may assume that $b_1 = d_1$, and so $a_1 < c_1$. Then X contains a negative or a nonpolarized gene. Let g_2 be such a gene of minimal rank $r(g_2) = k$. If i is even and $2 \leq i \leq k - 1$, we have $a_i - a_{i+1} = b_{i-1} - b_i = \cdots = a_0 - a_1 > c_0 - c_1 \geq c_i - c_{i+1}$. Hence $a_j < c_j$ for $1 \leq j \leq k$ and we have $X \rightarrow Z \leq Y$, where $X \rightarrow Z$ is defined by

$$\begin{aligned} g_1 + g_2 &\rightarrow g(1) + g^+(k+1) & \text{if } g_2 = g(k); \\ g_1 + g_2 &\rightarrow g(1) + g(k+1) & \text{if } g_2 = g^-(k). \end{aligned}$$

Case 5: $m = 1$. We have now $a_1 = b_1 < c_1 = d_1$. Therefore X contains a negative or a nonpolarized gene. Let g_2 be such a gene of minimal rank $r(g_2) = k$. If i is even and $2 \leq i \leq k$ then

$$a_{i-1} - a_i = b_{i-2} - b_{i-1} = \cdots = b_0 - b_1 > d_0 - d_1 \geq c_1 - c_2 \geq \cdots \geq c_{i-1} - c_i.$$

It follows that $a_j < c_j$ for $1 \leq j \leq k$. Then we have $X \rightarrow Z \leq Y$, where $X \rightarrow Z$ is defined by

$$\begin{aligned} g_1 + g_2 &\rightarrow g^+(k+1) & \text{if } g_2 = g(k); \\ g_1 + g_2 &\rightarrow g(k+1) & \text{if } g_2 = g^-(k). \end{aligned}$$

17. The cases $\Phi = (\Pi, 2\Lambda)$ or $(2\Lambda, \Pi)$. All mutations in this proof will be Φ -mutations. We treat the two cases simultaneously. Let $X, Y \in \Phi(n)$ and $X < Y$. We have to show that there exists a mutation $X \rightarrow Z$ such that $Z \leq Y$. We use induction on n . If $n = 0, 1$ then the assertion is vacuously true; so let $n > 1$. Let $(a_i, b_i) = \text{sig } X^{(i)}$ and $(c_i, d_i) = \text{sig } Y^{(i)}$, and let $r_i = a_i + b_i$, $s_i = c_i + d_i$. Note that $(a_i, b_i) \leq (c_i, d_i)$, $r_0 = n = s_0$, and $a_0 = c_0$, $b_0 = d_0$. Furthermore, if $\Phi = (2\Lambda, \Pi)$ then $a_i = b_i$ and $c_i = d_i$ for i odd, while if $\Phi = (\Pi, 2\Lambda)$ then $a_i = b_i$ and $c_i = d_i$ for i even.

As in §16 we may assume that X and Y are disjoint. In view of the lifting property we may also assume that

$$(17.1) \quad Y^{(i)} \neq 0 \text{ \& } i \geq 1 \Rightarrow r_i < s_i.$$

Assume that $X \supset 2g(m)$ for some m . Since X and Y are disjoint and $X < Y$, we have $Y^{(m)} \neq 0$ and by (17.1) $a_m < c_m$ or $b_m < d_m$. Define $X \rightarrow Z$ by $2g(m) \rightarrow g^\varepsilon(m+1) + g^{-\varepsilon}(m-1)$ where $\varepsilon = +$ if $a_m < c_m$ and $\varepsilon = -$ if $a_m = c_m$. Then $Z \leq Y$. Hence from now on we may assume (and we do) that X is polarized.

Hence all inequalities (15.2)–(15.7) are valid. By replacing each nonpolarized pair $2g(m)$ in Y by $g^+(m) + g^-(m)$ we obtain a polarization U of Y . Since $\text{sig } U^{(i)} = (c_i, d_i)$, it follows that the c_i 's and the d_i 's satisfy the analogs of the inequalities (15.2)–(15.7). In the sequel we shall use freely these inequalities without explicit reference to them.

Let us assume that $X \supset g^+(m) + g^-(m)$ for some m . Assume also that m is minimal with respect to this property. Since $Y^{(m)} \neq 0$ and s_m and r_m are even, it follows from (17.1) that $s_m - r_m \geq 2$.

If $X \supset 2g^+(m) + 2g^-(m)$ then $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $2g^+(m) + 2g^-(m) \rightarrow 2g(m-1) + 2g(m+1)$.

Now let $X \supset 2g^\epsilon(m) + g^{-\epsilon}(m)$ and $X \not\supset 2g^{-\epsilon}(m)$. By switching $+$ and $-$ signs in X and Y (if necessary) we may assume that $\epsilon = +$. If $Y^{(m+1)} \neq 0$ we can choose ϵ_1 so that $r^{\epsilon_1}(X^{(m+1)}) < r^{\epsilon_1}(Y^{(m+1)})$. Then $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $2g^+(m) + g^-(m) \rightarrow 2g(m-1) + g^{\epsilon_1}(m+2)$. Now let $Y^{(m+1)} = 0$. Since X and Y are disjoint and $X < Y$, we have $Y \supset 2g(m+1)$, and so $b_{m-1} = 1 < d_{m-1}$. Then $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $2g^+(m) + g^-(m) \rightarrow g^+(m-2) + 2g(m+1)$.

Now let $m \geq 2$, $X \not\supset 2g^+(m)$, and $X \not\supset 2g^-(m)$. Then $a_{m-1} < c_{m-1}$ or $b_{m-1} < d_{m-1}$. Without any loss of generality we may assume that $b_{m-1} < d_{m-1}$. If $b_{m+1} < d_{m+1}$ then $X \rightarrow Z \leq Y$ with $X \rightarrow Z$ defined by $g^+(m) + g^-(m) \rightarrow g^+(m-2) + g^-(m+2)$. Now let $b_{m+1} = d_{m+1}$. Then $a_{m-1} - a_m - 1 = b_m - b_{m+1} = b_m - d_{m+1} < d_m - d_{m+1} \leq c_{m-1} - c_m$, and so $c_{m-1} - a_{m-1} > (c_m - a_m) - 1 \geq 0$. If $a_{m+1} < c_{m+1}$ then $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $g^+(m) + g^-(m) \rightarrow g^-(m-2) + g^+(m+2)$. Now let also $a_{m+1} = c_{m+1}$. Then by (17.1) $Y^{(m+1)} = 0$, and so $Y \supset 2g(m+1)$. Let $g^\epsilon(k)$ be a gene of $X - g^+(m) - g^-(m)$ of maximal rank. Clearly $k < m$ and, by minimality of m , $X \not\supset g^{-\epsilon}(k)$. This implies that $k \geq 2$. Then we have $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $g^\epsilon(k) + g^+(m) + g^-(m) \rightarrow g^\epsilon(k-2) + 2g(m+1)$.

Finally let $m = 1$, $X \not\supset 2g^+(1)$, and $X \not\supset 2g^-(1)$. Then Y has no genes of rank one. Thus $X - g^+(1) - g^-(1)$ is nonempty, and let $g^\epsilon(k)$ be its gene of minimal rank. Without loss of generality we may assume (and we do) that $\epsilon = +$. We have

$$a_0 - a_2 = b_0 - b_2 = r_0 - r_1 - 1 > s_0 - s_1 = c_0 - c_2 = d_0 - d_2,$$

and so $a_2 < c_2$ and $b_2 < d_2$. If i is even and $2 \leq i \leq k-3$, then

$$\begin{aligned} a_i - a_{i+2} &= r_i - r_{i+1} = r_0 - r_1 - 2 \geq s_0 - s_1 \geq s_i - s_{i+1} \geq c_i - c_{i+2}, \\ b_i - b_{i+2} &= r_i - r_{i+1} \geq s_i - s_{i+1} \geq d_i - d_{i+2}, \end{aligned}$$

and consequently $a_j < c_j$ and $b_j < d_j$ for $1 \leq j \leq k$.

If $X \supset g^-(k)$ then we have $X \rightarrow Z \leq Y$, where $X \rightarrow Z$ is defined by $g^+(1) + g^-(1) + g^+(k) + g^-(k) \rightarrow 2g(k+1)$.

Otherwise we have

$$a_{k-1} - a_{k+1} = r_{k-1} - r_k = r_0 - r_1 - 2 \geq s_0 - s_1 \geq c_{k-1} - c_{k+1},$$

and so $a_{k+1} < c_{k+1}$. Now we have $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $g^+(1) + g^-(1) + g^+(k) \rightarrow g^+(k+2)$.

Hence from now on we may assume (and we do) that $X \not\supset g^+(m) + g^-(m)$, $m \geq 1$,

In the remaining part of the proof let m be the minimum rank of a gene of X . We assume (as we may) that $X \supset g^+(m)$, and consequently $X \not\supset g^-(m)$. We now distinguish several cases.

Case 1: $m \geq 3$. From $a_{m-1} - a_{m+1} = r_{m-1} - r_m = r_0 - r_1 > s_0 - s_1 \geq s_{m-1} - s_m \geq c_{m-1} - c_{m+1}$ and $b_{m-3} - b_{m-1} = r_0 - r_1 > s_0 - s_1 \geq d_{m-3} - d_{m-1}$ it follows that $a_{m+1} < c_{m+1}$ and $b_{m-1} < d_{m-1}$. If $X \supset 2g^+(m)$ then we have $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $2g^+(m) \rightarrow g^+(m-2) + g^+(m+2)$. Now assume that $X \not\supset 2g^+(m)$ and let $g^\epsilon(k)$ be a gene of $X - g^+(m)$ of minimum rank. Clearly

$k \geq m + 2$. If $i - m$ is odd and $m + 1 \leq i \leq k - 3$ then

$$a_i - a_{i+2} = r_i - r_{i+1} = r_0 - r_1 - 1 \geq s_0 - s_1 \geq s_i - s_{i+1} \geq c_i - c_{i+2}.$$

If $\varepsilon = +$ this is also valid for $i = k - 1$. Since $a_{m+1} < c_{m+1}$, we conclude that $a_j \leq c_j$ holds for $m \leq j \leq k$ and also for $j = k + 1$ if $\varepsilon = +$. If $i - m$ is odd and $m - 1 \leq i \leq k - 3$ then $b_i - b_{i+2} = r_0 - r_1 - 1 \geq s_0 - s_1 \geq d_i - d_{i+2}$. If $\varepsilon = -$ this is also valid for $i = k - 1$. Since $b_{m-1} < d_{m-1}$, we conclude that $b_j < d_j$ holds for $m - 1 \leq j \leq k$ and also for $j = k + 1$ if $\varepsilon = -$. Hence we have $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $g^+(m) + g^\varepsilon(k) \rightarrow g^+(m - 2) + g^\varepsilon(k + 2)$.

Case 2: $m = 2$. Since $a_1 - a_3 = r_1 - r_2 = r_0 - r_1 > s_0 - s_1 \geq s_1 - s_2 \geq c_1 - c_3$ and $a_1 \leq c_1$, we have $a_3 < c_3$. If $b_1 < d_1$ we can proceed exactly as in Case 1. Thus we may assume that $b_1 = d_1$, and consequently $a_1 < c_1$. It follows that $X \supset g^-(k)$ for some k . Clearly $k \geq 4$ and we may assume that k is minimal. If i is even and $2 \leq i \leq k - 2$ we have

$$a_i - a_{i+1} = b_{i-1} - b_i = \dots = a_0 - a_1 > c_0 - c_1 \geq c_i - c_{i+1},$$

and so $c_{i+1} - a_{i+1} > c_i - a_i \geq 1$. Hence $c_j - a_j \geq 2$ for j odd satisfying $3 \leq j \leq k - 1$. If $a_{k+1} < c_{k+1}$ then we have $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $g^+(2) + g^-(k) \rightarrow g^+(k + 2)$. Now let $a_{k+1} = c_{k+1}$. Since

$$b_{k-1} - b_k = a_0 - a_1 > c_0 - c_1 \geq c_k - c_{k+1} = c_k - a_{k+1} > a_k - a_{k+1},$$

we have $X \supset 2g^-(k)$. Now we have $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $g^+(2) + 2g^-(k) \rightarrow 2g(k + 1)$.

Case 3: $m = 1$ and $X \supset 2g^+(1)$. Since $a_1 = b_1 < c_1 = d_1$, we have $X \supset g^-(k)$ for some k . Assume that k is minimal; of course $k \geq 3$. If i is even and $2 \leq i \leq k - 1$ we have

$$a_{i-1} - a_i = b_{i-2} - b_{i-1} = \dots = b_0 - b_1 > d_0 - d_1 \geq c_1 - c_2 \geq \dots \geq c_{i-1} - c_i,$$

and so $c_i - a_i > c_{i-1} - a_{i-1} \geq 1$, $c_i - a_i \geq 2$. If $a_{k+1} < c_{k+1}$ then we have $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $2g^+(1) + g^-(k) \rightarrow g^+(k + 2)$. Now let $a_{k+1} = c_{k+1}$. Since $b_{k-1} - b_k = b_0 - b_1 > d_0 - d_1 \geq c_k - c_{k+1} = c_k - a_{k+1} > a_k - a_{k+1}$, we have $X \supset 2g^-(k)$. Thus we have $X \rightarrow Z \leq Y$ where $X \rightarrow Z$ is defined by $2g^+(1) + 2g^-(k) \rightarrow 2g(k + 1)$.

Case 4: $m = 1$ and $X \not\supset 2g^+(1)$. Since $s_1 - r_1 \geq 2$, we have $r_0 - r_1 \geq s_0 - s_1 + 2$. In particular, X has at least three genes. Let us write

$$X = g^+(1) + g^\varepsilon(k) + g^{\varepsilon_1}(t) + X_1, \quad 3 \leq k \leq t,$$

where $X_1 \in \Phi$ has no genes of rank less than t . We define a mutation $X \rightarrow Z$ by

$$g^\varepsilon(k) + g^{\varepsilon_1}(t) \rightarrow g^\varepsilon(k - 2) + g^{\varepsilon_1}(t + 2).$$

We have

$$\begin{aligned} a_{k-3} - a_{k-1} &= r_{k-3} - r_{k-2} \geq r_0 - r_1 - 1 > s_0 - s_1 \\ &\geq s_{k-3} - s_{k-2} \geq c_{k-3} - c_{k-1}, \\ b_{k-3} - b_{k-1} &= r_{k-2} - r_{k-1} = r_0 - r_1 - 1 > s_0 - s_1 \\ &\geq s_{k-3} - s_{k-2} \geq d_{k-3} - d_{k-1}, \end{aligned}$$

and so $a_{k-1} < c_{k-1}$ and $b_{k-1} < d_{k-1}$.

If $k = t$ we have $\epsilon_1 = \epsilon$. Then

$$\begin{aligned} r^\epsilon(X^{(k-1)} - X^{(k+1)}) &= r_{k-1} - r_k = r_0 - r_1 - 1 > s_0 - s_1 \\ &\geq s_{k-1} - s_k \geq r^\epsilon(Y^{(k-1)} - Y^{(k+1)}). \end{aligned}$$

Since $r^\epsilon(X^{(k-1)}) < r^\epsilon(Y^{(k-1)})$, we infer that also $r^\epsilon(X^{(k+1)}) < r^\epsilon(Y^{(k+1)})$ and so $Z \leq Y$.

Now let $k < t$. Let $i - k$ be odd and $k - 1 \leq i \leq t - 3$. Then

$$\begin{aligned} a_i - a_{i+2} &\geq r_k - r_{k+1} = r_0 - r_1 - 2 \geq s_0 - s_1 \geq s_i - s_{i+1} \geq c_i - c_{i+2}, \\ b_i - b_{i+2} &\geq r_k - r_{k+1} \geq d_i - d_{i+2}. \end{aligned}$$

If $\epsilon_1 = +$ then by the same argument we find that $a_{t-1} - a_{t+1} \geq c_{t-1} - c_{t+1}$, while if $\epsilon_1 = -$ we obtain $b_{t-1} - b_{t+1} \geq d_{t-1} - d_{t+1}$. Since $a_{k-1} < c_{k-1}$ and $b_{k-1} < d_{k-1}$, these inequalities imply that $Z \leq Y$.

This completes the proof of Theorem 6.

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